Math 201 :
Linear Algebra

Day 1 Welcome and warmup

Learning Goals

- Course structure and policies
- Course goals
- The big picture of linear algebra

See syllabus

Humble beginnings

2 variables

A linear equation looks like $5x - 2y = 3$.

A system of linear equations looks like

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + y + z &= -2 \\ x + 2y + z &= 2 \end{aligned}$$

3 equations, 3 variables

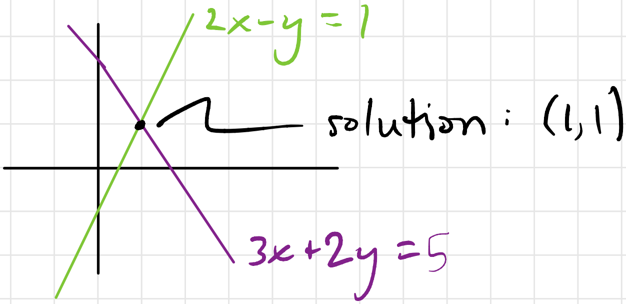
We want to solve systems of linear equations over a field F.

($F = \mathbb{R}$ (maybe $\mathbb{C}$) is standard, but we will care about $F = \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, \ldots$

as well, and the theory is the same (until we get to eigenvalues, etc.)

We will use matrices and row reduction algorithms to solve.

When $F = \mathbb{R}$, the solutions also have a geometry



Vector spaces and linear transformations

Make a conceptual leap by abstracting:

$$F^n \rightsquigarrow V \quad \text{vector space}$$

$$\cup| \qquad\qquad \cup|$$

$$\left\{\begin{matrix}\text{sol'ns to} \\ \text{homogeneous linear system}\end{matrix}\right\} \rightsquigarrow W \quad \text{subspace}$$

$$n \rightsquigarrow \text{dimension of } V$$

Vector spaces are related by linear transformations

$$L : V \longrightarrow W$$
$$v + w \longmapsto L(v+w) = L(v) + L(w), \quad v, w \in V$$
$$\lambda v \longmapsto L(\lambda v) = \lambda L(v), \quad \lambda \in F$$

By showing that every vector space has a basis (coordinates), we will construct a dictionary

matrices $\rightsquigarrow$ linear transformations.

Rank-nullity

with apologies for the violent nomenclature,

a linear transformation is essentially determined by what it kills and what it hits.

$$\ker(L) = \{v \in V \mid L(v) = 0\}$$
$$\mathrm{im}(L) = \{L(v) \mid v \in V\}$$

These are linear subspaces with dimensions and

$$\dim \ker(L) + \dim \mathrm{im}(L) = \dim V.$$

rank-nullity theorem

This is a powerful tool for computing characteristics of solution spaces!

Determinants

- When do n linear equations in n variables have a unique solution?
- When is a linear transformation $L: V \to V$ invertible?

Both phenomena are detected by the determinant. We will characterize determinants

- by universal property, and
- geometrically (over $\mathbb{R}$).

det is deep

Eigenstuff and diagonalization

$L: V \to V$ is a linear transformation and $v \in V \setminus \{0\}$ such that

$$L(v) = \lambda v$$

eigenvector

eigenvalue

If V has a basis of eigenvectors, then it can be represented by a diagonal matrix $\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$ — a Panglossian predicament.

We'll learn how to compute eigenstuff and diagonalizations (when they exist).

<u>And then some</u>

- Inner product spaces (length, angles)
- Applications — walks on graphs
  linear differential eq'ns
  linear recurrences
  ...