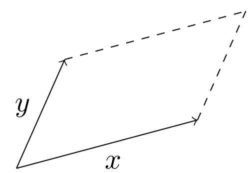
MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 14

Problem 1. A *rhombus* is a parallelogram with all sides of equal length. Using the standard inner product in $\mathbb{R}^2$, prove that a parallelogram is a rhombus if and only if its diagonals meet perpendicularly. Hint: take two arbitrary vectors $x, y \in \mathbb{R}^2$ and consider the parallelogram determined by x and y:



(Another hint: avoid coordinates. I.e., you should use abstract properties of the inner product and definitions of length and angle rather than formulas involving x_1, x_2, y_1, y_2.)

Problem 2. Let V be an n-dimensional vector space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\langle\, , \rangle$ be an inner product on V. Let $\alpha = \{v_1, \ldots, v_n\}$ be an ordered basis for V (not necessarily orthogonal). Let A be the $n \times n$ matrix given by

$$A_{ij} = \langle v_i, v_j \rangle.$$

For $x \in V$, let $[x]_\alpha \in F^n$ denote the coordinate vector for x with respect to the basis α. So if $x = \sum_{i=1}^n a_i v_i$, then $[x]_\alpha = (a_1, \ldots, a_n)$. (We have denoted this $\mathrm{Rep}_\alpha(x)$ in the past.) As usual, we will think of this vector in F^n as an $n \times 1$ matrix.

(a) Prove that for all $x, y \in V$,

$$\langle x, y \rangle = ([x]_\alpha)^\top A \left(\overline{[y]_\alpha} \right).$$

(Recall that for a matrix C, we define $\overline{C}$ by $\overline{C}_{ij} = \overline{C_{ij}}$, and then we define the conjugate transpose by $C^* = \overline{C^\top}$. *Hint*: compute both sides using sum notation. On the right-hand side, you will be computing the $1,1$-entry of a 1×1 matrix.)

(b) Prove that the matrix A satisfies $A = A^*$.

(c) If the basis α is orthonormal, what is the matrix A?

(d) (Extra credit) Let $\mathcal{D}$ be another ordered basis for V, and let C be the associated $n \times n$ matrix. How are A and C related (with proof)?

Problem 3. Let V be the vector space of all continuous functions $[0, 1] \to \mathbb{R}$ with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$. Let W be the subspace spanned by $\{t, \sqrt{t}\}$. (Warning: to get this problem right, you will need to be very careful with your calculations and double-check your solutions.

(a) Apply Gram-Schmidt to $\{t, \sqrt{t}\}$ to compute an orthonormal basis $\{u_1, u_2\}$ for W. (Hint: the coefficient of t in u_2 should be $-6\sqrt{2}$.)

(b) Find the closest function in W to $f(t) = t^2$. Express your solution in two forms: (i) as a linear combination of u_1 and u_2, and (ii) as a linear combination of t and $\sqrt{t}$.

(c) Graph f and its projection onto W (which you just calculated).