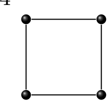
MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 13

Problem 1. Consider the *cycle graph* C_4:



(a) Find the adjacency matrix $A = A(G)$.
(b) Compute A^4 and use it to determine the number of walks from v_1 to v_3 of length 4. List all of these walks (these will be ordered lists of 5 vertices).
(c) What is the total number of *closed* walks of length 4?
(d) Compute and factor the characteristic polynomial for A.
(e) What are the algebraic multiplicities of each of the eigenvalues?
(f) Diagonalize A using our algorithm: compute bases for the eigenspaces of each of the eigenvalues you just found, and use them to construct a matrix P such that $P^{-1}AP$ is a diagonal matrix with the eigenvalues along the diagonal.
(g) Use part (f) to find a closed expression for A^ℓ for each $\ell \geq 1$. Use this expression to then give separate expressions for the cases where ℓ is even and where ℓ is odd.
(h) Use part (g) to take the trace of A^ℓ to get a formula for the number of closed walks of length ℓ for each $\ell \geq 1$.

Problem 2. In this exercise we will prove the theorem from class:

Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{N}$. Then then number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$.

(a) Let $p(i, j, \ell)$ denote the number of walks of length ℓ in G from v_i to v_j. Prove that for all $i, j = 1, \ldots, n$ and $\ell \geq 1$,

$$p(i, j, \ell) = \sum_{k=1}^{n} p(i, k, \ell - 1)p(k, j, 1).$$

(*Hint:* Part of the trick is to parse this formula appropriately.)
(b) Prove the theorem by induction on ℓ, using the result from part (a).

Problem 3. Suppose that

$$x' = Ax$$

for $x = (x_1(t), x_2(t))$ and $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

(a) Diagonalize A.
(b) Use your diagonalization to compute the exponential matrix e^{At}.
(c) Solve for x given the initial condition $x(0) = (1, 1)$.