MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 11

Problem 1. For each of the following matrices $A \in M_{n\times n}(F)$

(i) Determine all eigenvalues of A.
(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
(iii) If possible, find a basis for F^n consisting of eigenvectors of A.
(iv) If successful in finding such a basis, determine an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbb{R}$.

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = \mathbb{R}$.

(c) $A = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$ for $F = \mathbb{R}$.

(d) $A = \begin{pmatrix} 7 & -5 \\ 10 & -7 \end{pmatrix}$ for $F = \mathbb{C}$.

(e) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = \mathbb{R}$.

Problem 2. Let $V = \mathbb{R}[x]_{\leq 3}$, the vector space of polynomials of degree at most 3 with real coefficients. Let L denote the linear endomorphism

$$\begin{aligned} L\colon V &\longrightarrow V \\ p(x) &\longmapsto xp'(x) + p'(x). \end{aligned}$$

(You do not need to prove that L is linear, but you should know how to.) Find the eigenvalues of L and determine if V has a basis of eigenvectors of L. If L has such a basis, provide one (written as a set of polynomials), and if not, explain why not.

Problem 3. Let $f\colon V \to V$ be a linear transformation. For a positive integer m, we define f^m inductively as $f \circ f^{m-1}$. Prove that if λ is an eigenvalue for f, then λ^m is an eigenvalue for f^m.

Problem 4. Define $T\colon \mathsf{Mat}_{n\times n}(\mathbb{R}) \to \mathsf{Mat}_{n\times n}(\mathbb{R})$ by $T(A) = A^\top$ (the transpose of A).

(a) Show that the only eigenvalues of T are 1 and -1. (*Hint:* Problem 3 might help.)
(b) For $n = 2$, describe the eigenvectors corresponding to each eigenvalue.
(c) Find an ordered basis α for $\mathsf{Mat}_{2\times 2}(\mathbb{R})$ such that the matrix that represents T with respect to α is diagonal.
(d) Repeat part (b) for an arbitrary $n > 2$.
(e) Repeat part (c) for $\mathsf{Mat}_{n\times n}(\mathbb{R})$.