MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Let $\mathbb{R}[x]_{\leq n}$ be the vector space of polynomials in x of degree at most n with coefficients in $\mathbb{R}$. Let $f\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 2}$ and $g\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}^3$ be the linear transformations defined by

$$f(p(x)) = (3+x)p'(x) + 2p(x) \quad \text{and} \quad g(a+bx+cx^2) = (a+b, c, a-b).$$

Let $\alpha = \langle 1, x, x^2 \rangle$ and $\mathcal{E}$ be the standard ordered basis for $\mathbb{R}^3$.

(a) Compute the matrix representing f with respect to the basis α for both the domain and codomain.
(b) Compute the matrix representing g with respect to the bases α and $\mathcal{E}$.
(c) Compute the matrix representing $g \circ f$ with respect to the bases α and $\mathcal{E}$ using the general method presented in class (as you just did in parts (a) and (b)). Then use Theorem Three.IV.2.7 to verify your result by showing your answer to this part of the problem is an appropriate product of the matrices from (a) and (b).

Problem 2. Let V be a vector space over a field F. Recall that the *identity function* is the linear function $\mathrm{id}_V\colon V \to V$ by $\mathrm{id}_V(v) = v$ for all $v \in V$.

(a) Let V be a vector space of dimension n and let α be an ordered basis for V. Show that the matrix representing id_V with respect to the basis α for both the domain and the codomain is I_n (the $n \times n$ identity matrix).
(b) Let V and W be vector spaces of dimension n and let $f\colon V \to W$ be an isomorphism with inverse $f^{-1}\colon W \to V$. Let α and β be ordered bases for V and W, respectively. If A is the matrix representing f with respect to the bases α and β, what is the matrix for f^{-1} with respect to the bases β and α?
(c) Consider the linear transformation

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (3x+y, -x+4y).$$

Using part (b), prove that f is an isomorphism by exhibiting its inverse using matrix calculations. Write the inverse in the form $g(x,y) = (\text{blah}, \text{blah})$.

Problem 3. Fix F-vectors spaces V and W with ordered bases $\alpha = \langle v_1, \dots, v_n \rangle$ and $\beta = \langle w_1, \dots, w_m \rangle$. Let $\alpha^* = \langle v_1^*, \dots, v_n^* \rangle$ and $\beta^* = \langle w_1^*, \dots, w_m^* \rangle$ represent the dual ordered bases of V^* and W^*, respectively. Suppose $f\colon V \to W$ is a linear transformation. In Week 7 Problem 5(c), you proved that if $A = A_\alpha^\beta(f)$, then

$$A^\top = A_{\beta^*}^{\alpha^*}(f^*)$$

where $A^\top$ is the *transpose* of A (rows and columns swapped). You also proved in 5(b) that when $g\colon U \to V$ is a linear transformation, $(f \circ g)^* = g^* \circ f^*$. Use these observations to write a **short** proof that for all $A \in \mathsf{Mat}_{m\times n}(F)$ and $B \in \mathsf{Mat}_{\ell \times m}(F)$,

$$(BA)^\top = A^\top B^\top.$$

Problem 4. The *trace* of an $n \times n$ matrix A is the sum of its diagonal elements:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) If A and B are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. (Use the definition of matrix multiplication and summation notation in your proof.)
(b) If P is an invertible $n \times n$ matrix, prove that $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$.
(c) Consider the following ordered basis for $\mathcal{M}_{2\times 2}$:

$$\alpha = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

A rote check reveals that tr is a linear transformation. Assuming this fact, compute the matrix representing the trace function $\mathrm{tr}\colon \mathsf{Mat}_{2\times 2}(F) \to F$ with respect to α for the domain and with respect to the basis $\{1\}$ for the codomain.