MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 7

Problem 1. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute, if possible, the following. If it is not possible, explain why.

(a) AB, (b) $A(2B+C)$, (c) $A+C$,
(d) $(AB)D$, (e) $A(BD)$, (f) AD.

Problem 2. Let A and B be $m \times n$ matrices over F, and C an $n \times p$ matrix over F. Prove that $(A+B)C = AC + BC$. (This is called the *right distributivity* property.) Use summation notation in your proof.

Problem 3. For each of the following matrices, use the algorithm from class to determine whether it has an inverse, and if so, find the inverse. For this problem only, your Gauss-Jordan Reductionist Club membership is revoked and you must show all of the steps! (**Pointer:** as with many linear algebra problems, it's easy to make arithmetic mistakes, but it's also easy to check your answer!)

$$\text{(a)} \quad \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & 4 \end{pmatrix}.$$

Problem 4. Let $\mathbb{R}[x]_{\leq n}$ be the vector space of polynomials in x of degree at most n with coefficients in $\mathbb{R}$. Define

$$f\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$$

$$p(x) \mapsto \int_0^x p(t)\,dt.$$

Find the matrix representing f with respect to the bases $\{1, x, x^2\}$ for $\mathbb{R}[x]_{\leq 2}$ and $\{1, x, x^2, x^3\}$ for $\mathbb{R}[x]_{\leq 3}$.

Problem 5. Suppose $f\colon V \to W$ is a linear transformation. Define $f^*\colon W^* \to V^*$ by the rule

$$\phi \longmapsto f^*(\phi) = \phi \circ f.$$

(Recall that ϕ is a linear map $W \to F$, so $\phi \circ f$ makes sense as a linear map $V \to F$, *i.e.*, an element of V^*.)

(a) Prove that f^* is a linear transformation.
(b) Suppose $g\colon U \to V$ is a linear transformation. Prove that $(f \circ g)^* = g^* \circ f^*$. Also verify that $\mathrm{id}_V^* = \mathrm{id}_{V^*}$.[1]
(c) Suppose that V and W have ordered bases $\langle v_1, \ldots, v_n \rangle$ and $\langle w_1, \ldots, w_m \rangle$, respectively, and suppose that f has matrix $A \in \mathsf{Mat}_{m \times n}(F)$ with respect to these ordered bases. What is the matrix of f^* relative to $\langle w_1^*, \ldots, w_m^* \rangle$ and $\langle v_1^*, \ldots, v_n^* \rangle$? (Your answer should be a matrix with entries expressed in terms of the entries of A.)

[1] This makes $(\ )^*$ a *constravariant functor*, but you needn't know what that means.