MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 6

Problem 1. For the following functions f:

(i) prove that f is a linear transformation,
(ii) find bases for $\ker(f)$ and $\mathrm{im}(f)$, and
(iii) compute the nullity and the rank.

(a) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x - y, 2z)$.
(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x, y) = (x + y, 0, 2x - y)$.
(c) $f\colon \mathbb{R}[x]_{\leq 2} \to \mathbb{R}[x]_{\leq 3}$ defined by $f(p(x)) = x \cdot p(x) + p'(x)$. Here, $p'(x)$ denotes the usual derivative from one-variable calculus.

(Recall that $F[x]_{\leq n}$ denotes the vector space of polynomials with coefficients in F of degree less than or equal to n. One basis for it is $\{1, x, x^2, \ldots, x^n\}$, and hence, it has dimension $n + 1$.)

Problem 2. Let V and W be vector spaces over F, and let $f\colon V \to W$ be a linear transformation.

(a) Prove that f is injective if and only if f carries linearly independent subsets of V to linearly independent subsets of W.
(b) Suppose that f is injective and that S is a subset of V. Prove that S is linearly independent if and only if $f(S)$ is linearly independent.

Problem 3. Fix an F-vector space V. Recall from Problem 5 of the Week 5 homework that $V^* = \mathrm{Hom}(V, F)$ is the *dual* of V. We write $V^{**} := (V^*)^*$ for the *double dual* of V; its elements are linear transformations from V^* to F. The *evaluation* map $\mathrm{ev}\colon V \to V^{**}$ is the function taking $v \in V$ to $(\mathrm{ev}_v\colon V^* \to F) \in V^{**}$ where $\mathrm{ev}_v(f) = f(v)$.

(a) Prove that ev is a linear transformations.
(b) It is a fact that for any nonzero $v \in V$, there exists $f \in V^*$ such that $f(v) \neq 0$. (*Challenge*: Prove it. If V is infinite dimensional, you will need to invoke the axiom of choice.) Use this to prove that ev is injective.
(c) Use Problem 5 of Week 5 to show that $V \cong V^* \cong V^{**}$ when V is finite dimensional.[1]

Use the following information in Problems 4 and 5:

Let V be a vector space over F. An *affine subspace* of V is a subset of the form

$$A = p + U := \{p + u : u \in U\} \tag{1}$$

where $p \in V$ and U is a linear subspace of V. The *dimension* of A is the dimension of its linear part: $\dim A := \dim U$. If $\dim A = k$, we call A a *k-plane* in V. A 1-plane is a *line*, and a 2-plane is simply called a *plane*. A $(\dim V - 1)$-plane is a *hyperplane*.

To give a *parametrization* of an affine subspace $A = p + U$ of V, first choose a basis $\{u_1, \ldots, u_k\}$ of U, and then define

$$\begin{aligned} \ell\colon F^k &\to V \\ (t_1, \ldots, t_n) &\mapsto p + t_1 u_1 + \cdots + t_k u_k. \end{aligned}$$

[1]When V is infinitely dimensional, this no longer holds. Also note that $V \cong V^*$ depends on the choice of a basis while the definition of ev does not depend on a basis; it is *canonical*.

Then ℓ is an injection whose image is A.

Problem 4. The expression of an affine subspace A as $p + U$ is not unique. Show that

$$p + U = q + U$$

if and only if

$$p - q \in U.$$

(Note: Make sure to use the definition of $p + U$ displayed in (1), above. Do not just subtract q from both sides of (1). Recall from Math 112 that in order to show two sets X and Y are equal, take $x \in X$ and show it must be in Y, then take $y \in Y$ and show it is in X.)

Problem 5. Let H be the plane in $\mathbb{R}^3$ containing the points $(1, 0, 0)$, $(4, 1, -2)$, and $(6, 1, 1)$.

(a) Find a linear equation whose solution set is H. Show your work.

(b) Express H as an affine subspace of $\mathbb{R}^3$ by finding $p \in \mathbb{R}^3$ and a linearly independent set $S \subset \mathbb{R}^3$ such that $H = p + \operatorname{span}(S)$.

(c) Give a parametrization of H.