MATH 201: LINEAR ALGEBRA
HOMEWORK DUE FRIDAY WEEK 3

Make sure to review the homework instructions in the syllabus before writing your solutions. In particular, show your work and write in complete sentences (but also aim for concise explanations).

Problem 1. In each of the following:

- » Determine whether the given vector v is in the span of the set S by creating a relevant system of linear equations in the usual form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

  and then row reducing the corresponding augmented matrix for the system.
- » If v is in the span of S, then explicitly write v as a linear combination of the vectors in S.

Assume we are working over the field $\mathbb{Q}$ of rational numbers.

(a) $v = (0, -1, -6)$, $S = \{(1, 0, -1), (2, 1, 3), (4, 2, 5)\}$.
(b) $v = (1, 2, 4)$, $S = \{(1, 4, 7), (2, 5, 8), (3, 6, 9)\}$.
(c) $v = x^3 - 13x^2 + 7x + 27$, $S = \{x^3 + 3x^2 - 2, x^3 + x^2 + 4x + 1, 2x^2 + x + 4\}$.
(d) $v = \begin{pmatrix} 9 & 12 \\ 10 & 9 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \right\}$.

Problem 2. Determine whether the following sets are linearly dependent or linearly independent.

(a) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $\mathbb{R}[x]$.
(b) $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in $\mathbb{R}^3$.
(c) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $\mathbb{R}^3$.
(d) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $(\mathbb{Z}/2\mathbb{Z})^3$ (where $\mathbb{Z}/2\mathbb{Z}$ is the field with two elements).

Problem 3. Let V be a vector space over $\mathbb{R}$. Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

Problem 4. Answer the following with "True" or "False". No justifications or counterexamples are required.

(a) If S is a linearly dependent set, then every vector in S is a linear combination of the other vectors in S.
(b) Any set containing the 0 vector is linearly dependent.
(c) Every subset of a linearly dependent set is linearly dependent.

Problem 5. Let F be a finite field with $|F| = q$.[1] Let $S = \{u_1, \ldots, u_n\}$ be a set of linearly independent vectors in an F-vector space. Determine the cardinality of $\operatorname{span} S$.

[1] Here $|F|$ denotes the *cardinality* of F, *i.e.*, the number of elements in the set F. In abstract algebra, you will discover that q must be of the form p^n for some prime p.

Problem 6. Fix a field F. Using only the material we have developed in class thus far, show that a set

$$S = \{(a,b),(c,d)\} \subseteq F^2$$

is linearly independent if and only if

$$ad - bc \neq 0.$$