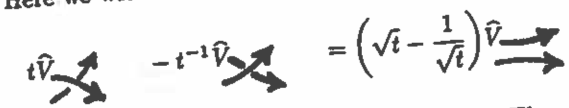
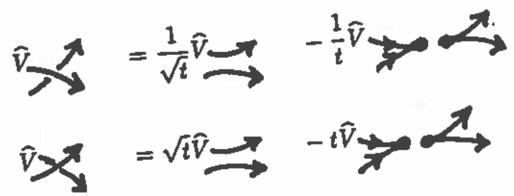
The rest proceeds in the same way.

Thus the model is an ambient isotopy invariant, but regular isotopy patterns of calculation live inside it!

**Remark.** It is sometimes convenient to use another parameterization of the Jones polynomial. Here we write



Note that the loop variable $\hat{\delta} = (\sqrt{t} + \frac{1}{\sqrt{t}})$ for this model. The expansion is given by the formulas:



In the corresponding state expansion we have

$$\hat{V}_K = \sum_{\vec{\sigma}} \langle K|\vec{\sigma}\rangle \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{\|\vec{\sigma}\|}$$

where $\langle K|\vec{\sigma}\rangle$ is the product of the vertex weights $(\sqrt{t}, 1/\sqrt{t}, -t, -1/t)$ and $\|\vec{\sigma}\|$ is the number of loops in the oriented state. For each state, the sign is $(-1)^{\pi}$ where $\pi$ is the parity of the number of creation-annihilation splices in the state. This version of the Jones polynomial may lead some insight into the vexing problem of cancellations in the state summation.

**Unknot Problem.** For a knot $K$, does $V_K = 1$ $(\hat{V}_K = 1)$ imply that $K$ is ambient isotopic to the unknot?



## $7^0$. Braids and the Jones Polynomial.

In this section I shall demonstrate that the normalized bracket $\mathcal{L}_K(A) = (-A^3)^{-w(K)}\langle K\rangle$ is a version of the original Jones polynomial $V_K(t)$ by way of the theory of braids. The Jones polynomial has been subjected to extraordinary generalizations since it was first introduced in 1984 [JO2]. These generalizations will emerge in the course of discussion. Here we stay with the story of the original Jones polynomial and its relation with the bracket.

Jones constructed the invariant $V_K(t)$ by a route involving braid groups and von Neumann algebras. Although there is much more to say about von Neumann algebras, it is sufficient here to consider a sequence of algebras $A_n$ $(n = 2, 3, \dots)$ with multiplicative generators $e_1, e_2, \dots, e_{n-1}$ and relations:

1) $e_i^2 = e_i$
2) $e_i e_{i\pm1} e_i = \tau e_i$
3) $e_i e_j = e_j e_i \qquad |i - j| > 2$

($\tau$ is a scalar, commuting with all the other elements.) For our purposes we can let $A_n$ be the free additive algebra on these generators viewed as a module over the ring $\mathbf{C}[\tau, \tau^{-1}]$ ($\mathbf{C}$ denotes the complex numbers.). The scalar $\tau$ is often taken to be a complex number, but for our purposes is another algebraic variable commuting with the $e_i$'s. This algebra arose in the theory of classification of von Neumann algebras [JO7], and it can itself be construed as a von Neumann algebra.

In this von Neumann algebra context it is natural to study a certain tower of algebras associated with an inclusion of algebras $N \subset M$. With $M_0 = N$, $M_1 = M$ one forms $M_2 = \langle M_1, e_1\rangle$ where $e_1 : M_1 \to M_0$ is projection to $M_0$ and $\langle M_1, e_1\rangle$ denotes an algebra generated by $M_1$ with $e_1$ adjoined. Thus we have the pattern

$$M_0 \subset M_1 \subset M_2 = \langle M_1, e_1\rangle, \qquad e_1 : M_1 \to M_0, \quad e_1^2 = e_1.$$

This pattern can be iterated to form a tower

$$M_0 \subset M_1 \subset M_2 \subset M_3 \subset \dots \subset M_n \subset M_{n+1} \subset \dots$$

with $e_i : M_i \to M_{i-1}$, $e_i^2 = e_i$ and $M_{i+1} = \langle M_i, e_i\rangle$. Jones constructs such a tower of algebras with the property that $e_i e_{i\pm1} e_i = \tau e_i$ and $e_i e_j = e_j e_i$ for $|i - j| > 1$.

Here he found that $\tau^{-1} = [M_1 : M_0]$ a generalized notion of index for these algebras. Furthermore, he defined a trace $\mathrm{tr} : M_n \to \mathbf{C}$ (complex numbers) such that it satisfied the

$$\boxed{\begin{array}{c}\textbf{Markov Property: } \mathrm{tr}(we_i) = \tau\ \mathrm{tr}(w) \\ \text{for } w \text{ in the algebra generated by } M_0, e_1, \ldots, e_{i-1}\end{array}}.$$

A function from an algebra $\mathcal{A}$ to the complex numbers is said to be a **trace** (tr) if it satisfies the identity

$$\mathrm{tr}(ab) = \mathrm{tr}(ba)$$

for $a, b, \in \mathcal{A}$. Thus ordinary matrix traces are examples of trace functions.

The **tower construction** is very useful for studying the index $[M_1 : M_0]$ for special types of von Neumann algebras. While I shall not discuss von Neumann algebras in this short course, we shall construct a model of such a tower that is directly connected with the bracket polynomial. Thus the combinatorial structure of the tower construction will become apparent from the discussion that follows.

Now to return to the story of the Jones polynomial: Jones was struck by the analogy between the relations for the algebra $A_n$ and the generating relations for the $n$-**strand Artin braid group** $B_n$. View Figure 11 for a comparison of these sets of relations

$$A_n$$

$$e_i^2 = e_i$$
$$e_i e_{i\pm1} e_i = \tau e_i$$
$$e_i e_j = e_j e_i, |i - j| > 1$$
$$e_1, e_2, \cdots, e_{n-1}$$

Jones Algebra

$$B_n$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i - j| > 1$$
$$\sigma_1^{\pm1}, \cdots, \sigma_{n-1}^{\pm1}$$

Artin Braid Group

**Figure 11**

Jones constructed a representation $\rho_n : B_n \to A_n$ of the Artin Braid group to the algebra $A_n$. The representation has the form

$$\rho_n(\sigma_i) = ae_i + b$$



with $a$ and $b$ chosen appropriately. Since $A_n$ has a trace $\mathrm{tr} : A_n \to \mathbf{C}[t, t^{-1}]$ one can obtain a mapping $\mathrm{tr} \circ \rho : B_n \to \mathbf{C}[t, t^{-1}]$. Upon appropriate normalization this mapping is the Jones polynomial $V_K(t)$. It is an ambient isotopy invariant for oriented links. While the polynomial $V_K(t)$ was originally defined only for braids, it follows from the theorems of Markov (see [B2]) and Alexander [ALEX1] that (due to the Markov property of the Jones trace) it is well-defined for arbitrary knots and links.

These results of Markov and Alexander are worth remarking upon here. First of all there is **Alexander's Theorem: Each link in three-dimensional space is ambient isotopic to a link in the form of a closed braid.**

**Braids.** A braid is formed by taking $n$ points in a plane and attaching strands to these points so that parallel planes intersect the strands in $n$ points. It is usually assumed that the braid begins and terminates in the same arrangement of points so that it has the diagrammatic form

$b$

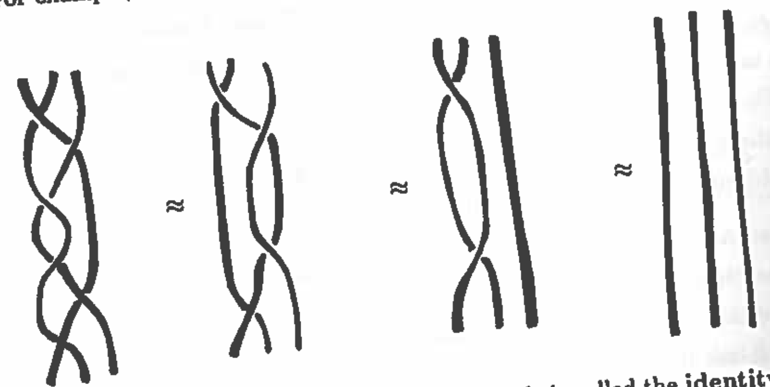
$\bar{b}$

Here I have illustrated a 3-strand braid $b \in B_3$, and its closure $\bar{b}$. The closure $\bar{b}$ of a braid $b$ is obtained by connecting the initial points to the end-points by a collection of parallel strands.

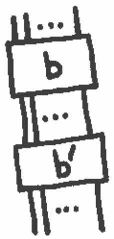
It is interesting and appropriate to think of the braid as a diagram of a physical process of particles interacting or moving about in the plane. In the diagram, we take the arrow of time as moving up the page. Each plane (spatial plane) intersects the page perpendicularly in a horizontal line. Thus successive slices give a picture of the motions of the particles whose world-lines sweep out the braid.

Of course, from the topological point of view one wants to regard the braid as a purely spatial weave of descending strands that are fixed at the top and the bottom of the braid. Two braids in $B_n$ are said to be equivalent (and we write $b = b'$ for this equivalence) if there is an ambient isotopy from $b$ to $b'$ that keeps the end-points fixed and does not move any strands outside the space between the top and bottom planes of the braids. (It is assumed that $b$ and $b'$ have identical input and output points.)

For example, we see the following equivalence




The braid consisting in $n$ parallel descending strands is called the **identity braid** in $B_n$ and is denoted by 1 or $1_n$ if need be. $B_n$, the collection of $n$-strand braids, up to equivalence, (i.e. the set of equivalence classes of $n$-strand braids) is a group - the **Artin Braid Group**. Two braids $b$, $b'$ are multiplied by joining the output strands of $b$ to the input strands of $b'$ as indicated below:




$bb'$


$bb'$

Every braid can be written as a product of the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and their inverses $\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_{n-1}^{-1}$. These elementary braids $\sigma_i$ and $\sigma_i^{-1}$ are obtained by interchanging only the $i$-th and $(i+1)$-th points in the row of inputs



Thus

$\sigma_1$ $\quad$ $\sigma_2$ $\quad \dots \quad$ $\sigma_{n-1}$

$\sigma_1^{-1}$ $\quad$ $\sigma_2^{-1}$ $\quad \dots \quad$ $\sigma_{n-1}^{-1}$ .

These generators provide a convenient way to catalog various weaving patterns. For example

$$= \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$$

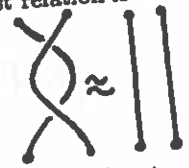
A 360° twist in the strands has the appearance

$$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$$
$$\|$$
$$(\sigma_1\sigma_2)^3.$$

The braid group $B_n$ is completely described by these generators and relations. The relations are as follows:

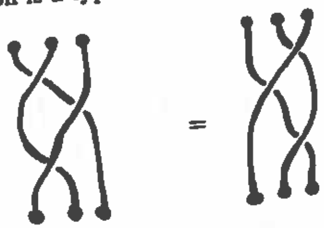
$$\begin{cases} \sigma_i\sigma_i^{-1} = 1, & i = 1, \dots, n-1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1, \dots, n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1. \end{cases}$$

Note that the first relation is a version of the type II move,



$\sigma_1\sigma_1^{-1} = 1$

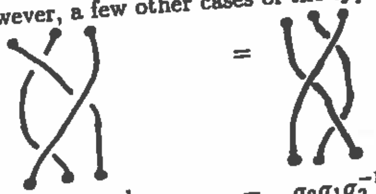
while the second relation is a type III move:



$$\sigma_1\sigma_2\sigma_1 \quad = \quad \sigma_2\sigma_1\sigma_2$$

Note that since $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is stated in the group $B_3$, we also know that $(\sigma_1\sigma_2\sigma_1)^{-1} = (\sigma_2\sigma_1\sigma_2)^{-1}$, whence $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$.

There are, however, a few other cases of the type III move. For example:



$$\sigma_1^{-1}\sigma_2\sigma_1 \quad = \quad \sigma_2\sigma_1\sigma_2^{-1}$$

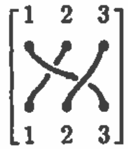
However, this is algebraically equivalent to the relation $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$ (multiply both sides by $\sigma_1$ on the left, and $\sigma_2$ on the right). In fact, we can proceed directly as follows:

$$\begin{aligned}\sigma_1^{-1}\sigma_2\sigma_1 &= \sigma_1^{-1}(\sigma_2\sigma_1\sigma_2)\sigma_2^{-1}\\ &= \sigma_1^{-1}(\sigma_1\sigma_2\sigma_1)\sigma_2^{-1}\\ &= (\sigma_1^{-1}\sigma_1)(\sigma_2\sigma_1\sigma_2^{-1})\\ &= \sigma_2\sigma_1\sigma_2^{-1}.\end{aligned}$$

I emphasize this form of the equivalence because it shows that the type III move with a mixture of positive and negative crossings can be accomplished via a combination of type II moves and type III moves where all the crossings have the same sign.



Note that there is a homomorphism $\pi$ of the braid group $B_n$ onto the permutation group $S_n$ on the set $\{1, 2, \ldots, n\}$. The map $\pi : B_n \to S_n$ is defined by taking the permutation of top to bottom rows of points afforded by the braid. Thus

$\pi$ 

$= \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

where the notation on the right indicates a permutation $\rho : \{1,2,3\} \to \{1,2,3\}$ with $\rho(1) = 3$, $\rho(2) = 1$, $\rho(3) = 2$. If $\rho = \pi(b)$ for a braid $b$, then $\rho(i) = j$ where $j$ is the lower endpoint of the braid strand that begins at point $i$.

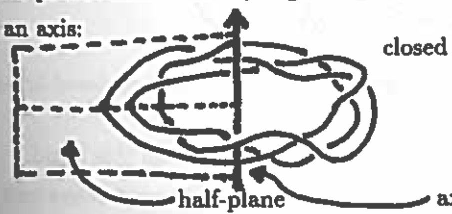
Letting $\tau_k : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ denote the transposition of $k$ and $k+1$: $\tau_k(i) = i$ if $i \neq k,\ k+1$, $\tau_k(k) = k+1$, $\tau_k(k+1) = k$. We have that $\pi(\sigma_i) = \tau_i$, $i = 1, \ldots, n-1$. In terms of these transpositions, $S_n$ has the presentation

$$S_n = (\tau_1, \ldots, \tau_{n-1} \mid \tau_i^2 = 1,\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}).$$

The permutation group is the quotient of the braid group $B_n$, obtained by setting the squares of all the generators equal to the identity.

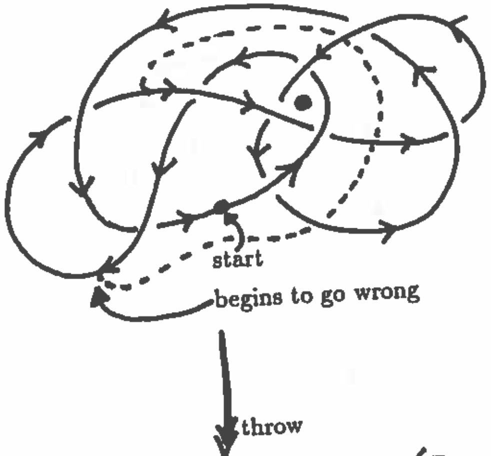
**Alexander's Theorem.**

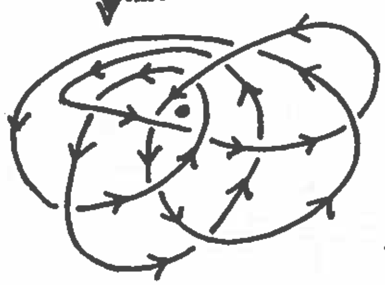
As we mentioned a few paragraphs ago, Alexander proved [ALEX1] that any knot or link could be put in the form of a closed braid (via ambient isotopy). Alexander proved this result by regarding a closed braid as a looping of the knot around an axis:




Thinking of three-dimensional space as a union of half-planes, each sharing the axis, we require that $\bar{b}$ intersect each half-plane in the same number of points – (the number of braid-strands). As you move along the knot or link, you are circulating the axis in either a clockwise or counterclockwise orientation. Alexander's method was to choose a **proposed braid axis.** Then follow along the knot or link, throwing the strand over the axis whenever it began to circulate incorrectly. Eventually, you have the link in braid form.

Figure 12 illustrates this process for a particular choice of axis. Note that it is clear that this process will not always produce the most efficient braid representation for a given knot or link. In the example of Figure 12 we would have fared considerably better if we had taken the axis at a different location - as shown below.

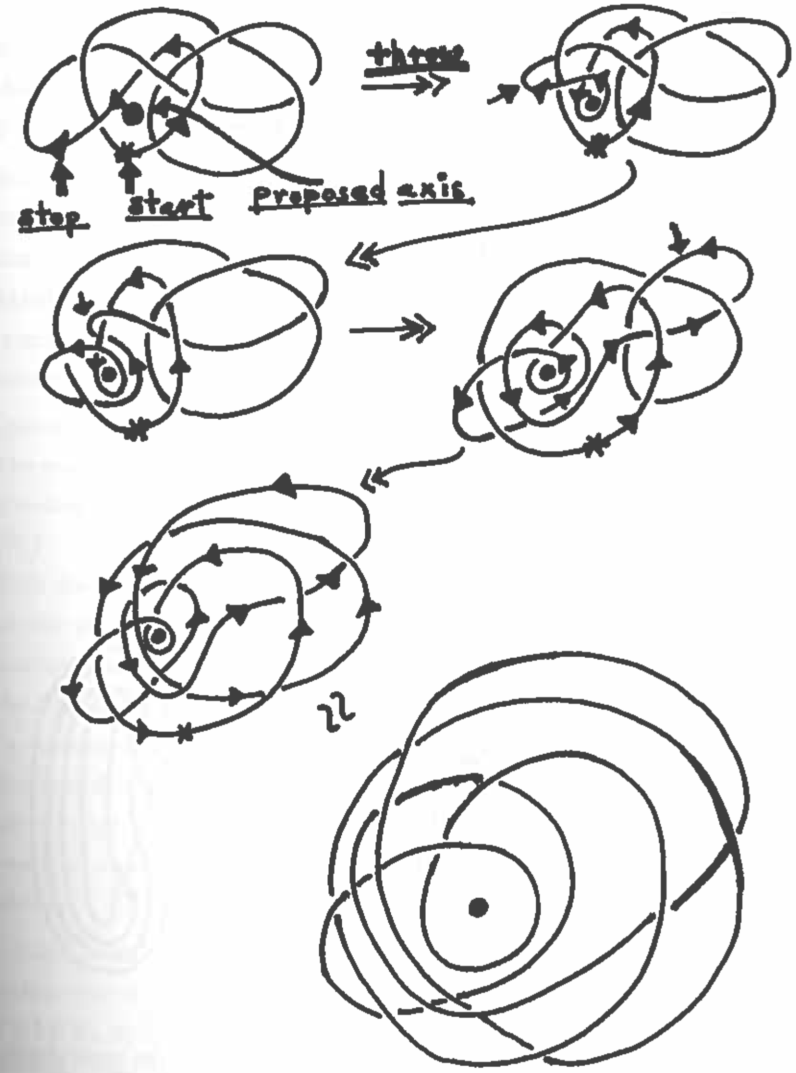



One throw over the new axis is all that is required to obtain this braid.

These examples raise the question: **How many different ways can a link be represented as a closed braid?**






Alexander's Theorem

**Figure 12**

There are some simple ways to modify braids so that their closures are ambient isotopic links. First there is the **Markov move**: Suppose $\beta$ is a braid word in $B_n$ (hence a word in $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and their inverses). Then the three braids $\beta$, $\beta\sigma_n$ and $\beta\sigma_n^{-1}$ all have ambient isotopic closures. For example,

$\beta$ =

$= \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2 \in B_3$

$\beta\sigma_3$ =

$\in B_4$

$\overline{\beta}$ =

$\overline{\beta\sigma_3}$ =

Thus $\overline{\beta\sigma_n^{\pm 1}}$ is obtained from $\beta$ by a type I Reidemeister move.

A somewhat more diabolical way to make a braid with the same closure is to choose any braid $g$ in $B_n$ and take the **conjugate braid** $g\beta g^{-1}$. When we close $g\beta g^{-1}$ to form $\overline{g\beta g^{-1}}$ the braid $g$ and its inverse $g^{-1}$ can cancel each other out by interacting through the closure strands. The fundamental theorem that relates the theory of knots and the theory of braids is the



**Markov Theorem 7.1.** Let $\beta_n \in B_n$ and $\beta'_m \in B_m$ be two braids in the braid groups $B_n$ and $B_m$ respectively. Then the links (closures of the braids $\beta$, $\beta'$) $L = \overline{\beta_n}$ and $L' = \overline{\beta'_m}$ are ambient isotopic if and only if $\beta'_m$ can be obtained from $\beta_n$ by a series of

1) equivalences in a given braid group.
2) conjugation in a given braid group. (That is, replace a braid by some conjugate of that braid.)
3) Markov moves: (A Markov move replaces $\beta \in B_n$ by $\beta\sigma_n^{\pm 1} \in B_{n+1}$ or the inverse of this operation - replacing $\beta\sigma_n^{\pm 1} \in B_{n+1}$ by $\beta \in B_n$ if $\beta$ has no occurrence of $\sigma_n$.)

For a proof of the Markov theorem the reader may wish to consult [B2].

The reader may enjoy pondering the question: How can Alexander's technique for converting links to braids be done in an algorithm that a computer can perform? (See [V].)

With the Markov theorem, we are in possession of the information needed **to use the presentations of the braid groups $B_n$ to extract topological information about knots and links.** In particular, it is now possible to explain how the Jones polynomial works in relation to braids. For suppose that we are given a commutative ring $R$ (polynomials or Laurent polynomials for example), and functions $J_n : B_n \to R$ from the $n$-strand braid group to the ring $R$, defined for each $n = 2, 3, 4, \ldots$ Then the Markov theorem assures us that the family of functions $\{J_n\}$ can be used to construct link invariants if the following conditions are satisfied:

1. If $b$ and $b'$ are equivalent braid words, then $J_n(b) = J_n(b')$. (This is just another way of saying that $J_n$ is well-defined on $B_n$.)
2. If $g, b \in B_n$ then $J_n(b) = J_n(gbg^{-1})$.
3. If $b \in B_n$, then there is a constant $\alpha \in R$, independent of $n$, such that

$$J_{n+1}(b\sigma_n) = \alpha^{+1} J_n(b)$$
$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b).$$

We see that for the closed braid $\overline{b'} = \overline{b\sigma_n}$ the result of the Markov move $b \mapsto b'$ is to perform a type I move on $\overline{b}$. Furthermore, $\overline{b\sigma_n}$ corresponds to a type I move of positive type, while $\overline{b\sigma_n^{-1}}$ corresponds to a type I move of negative type. It is for this reason that I have chosen the conventions for $\alpha$ and $\alpha^{-1}$ as above. Note also that, orienting a braid downwards, as in

$\sigma_n$

has positive crossings corresponding to $\sigma_i$'s with positive exponents.

With these remarks in mind, let's define the **writhe of a braid**, $w(b)$, to be its **exponent sum**. That is, we let $w(b) = \sum_{\ell=1}^{k} a_\ell$ in any braid word

$$\sigma_{i_1}^{a_1}\sigma_{i_2}^{a_2}\cdots\sigma_{i_k}^{a_k}$$

representing $b$. From our previous discussion of the writhe, it is clear that $w(b) = w(\bar{b})$ where $\bar{b}$ is the oriented link obtained by closing the braid $b$ (with downward-oriented strands). Here $w(\bar{b})$ is the writhe of the oriented link $\bar{b}$.

**Definition 7.2.** Let $\{J_n : B_n \to R\}$ be given with properties 1., 2., 3. as listed above. Call $\{J_n\}$ a **Markov trace** on $\{B_n\}$. For any link $L$, let $L \sim \bar{b}$, $b \in B_n$ via Alexander's theorem. Define $J(L) \in R$ via the formula

$$J(L) = \alpha^{-w(b)} J_n(b).$$

Call $J(L)$ the **link invariant for the Markov trace** $\{J_n\}$.

**Proposition 7.3.** Let $J$ be the link invariant corresponding to the Markov trace $\{J_n\}$. Then $J$ is an invariant of ambient isotopy for oriented links. That is, if $L \sim L'$ ($\sim$ denotes ambient isotopy) then $J(L) = J(L')$.

**Proof.** Suppose, by Alexander's theorem, that $L \sim \bar{b}$ and $L' \sim \bar{b'}$ where $b \in B_n$ and $b' \in B_n$ are specific braids. Since $L$ and $L'$ are ambient isotopic, it follows that $\bar{b}$ and $\bar{b'}$ are also ambient isotopic. Hence $\bar{b'}$ can be obtained from $\bar{b}$ by a sequence of Markov moves of the type 1., 2., 3. Each such move leaves the function $\alpha^{-w(b)}J_n(b)$ ($b \in B_n$) invariant since the exponent sum is invariant under conjugation, braid moves, and it is used here to cancel the effect of the type 3. Markov move. This completes the proof. //

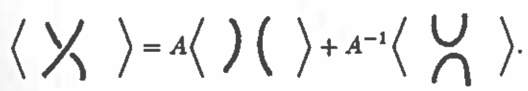


**The Bracket for Braids.**

Having discussed generalities about braids, we can now look directly at the bracket polynomial on closed braids. In the process, the structure of the Jones polynomial and its associated representations of the braid groups will naturally emerge.

In order to begin this discussion, let's define $\langle\ \rangle : B_n \to \mathbf{Z}[A, A^{-1}]$ via $\langle b\rangle = \langle\bar{b}\rangle$, the evaluation of the bracket on the closed braid $b$. In terms of the Markov trace formalism, I am letting $J_n : B_n \to \mathbf{Z}[A, A^{-1}] = R$ via $J_n(b) = \langle\bar{b}\rangle$. In fact, given what we know about the bracket from section $3^0$, it is obvious that $\{J_n\}$ is a Markov trace, with $\alpha = -A^3$.

Now consider the **states of a braid**. That is, consider the states determined by the recursion formula for the bracket:



In terms of braids this becomes

$$\langle\sigma_i\rangle = A\langle 1_n\rangle + A^{-1}\langle U_i\rangle$$

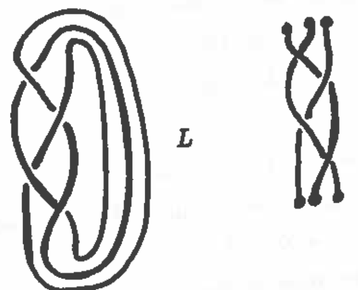
where $1_n$ denotes the identity element in $B_n$ (henceforth denoted by 1), and $U_i$ is a new element written in braid input-output form, but with a cup-∪ cap-∩ combination  at the $i$-th and $(i+1)$-th strands:

 ,  ,  (for 4-strands)

$U_1$ $U_2$ $U_3$

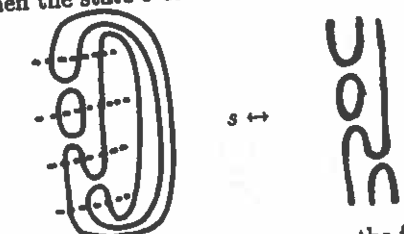
Since a state for $\bar{b}$, is obtained by choosing splice direction for each crossing of $b$, we see that **each state of $\bar{b}$ can be written as the closure of an (input-output) product of the elements $U_i$.** (See section $3^0$ for a discussion of bracket states.)

For example, let $L = \bar{b}$ be the link




$b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

Then the state $s$ of $L$ shown below corresponds to the product $U_1^2U_2$.




$\overline{U_1^2U_2} = s$

In fact it is clear that we can use the following formalism: Write

$$\sigma_i \equiv A + A^{-1}U_i, \qquad \sigma_i^{-1} \equiv A^{-1} + AU_i$$

Given a braid word $b$, write $b \equiv \mathcal{U}(b)$ where $\mathcal{U}(b)$ is a sum of products of the $U_i$'s, obtained by performing the above substitutions for each $\sigma_i$. Each product of $U_i$'s, when closed gives a collection of loops. Thus if $U$ is such a product, then $\langle U\rangle = \langle\overline{U}\rangle = \delta^{\|U\|}$ where $\|U\| = \#(\text{of loops in})\ \overline{U} - 1$ and $\delta = -A^2 - A^{-2}$. Finally if $\mathcal{U}(b)$ is given by

$$\mathcal{U}(b) = \sum_s \langle b|s\rangle U_s$$

where $s$ indexes all the terms in the product, and $\langle b|s\rangle$ is the product of $A$'s and $A^{-1}$'s multiplying each $U$-product $U_s$, then

$$\langle b\rangle = \langle\mathcal{U}(b)\rangle = \sum_s \langle b|s\rangle\ \langle U_s\rangle$$

$$\langle b\rangle = \sum_s \langle b|s\rangle\delta^{\|s\|}.$$

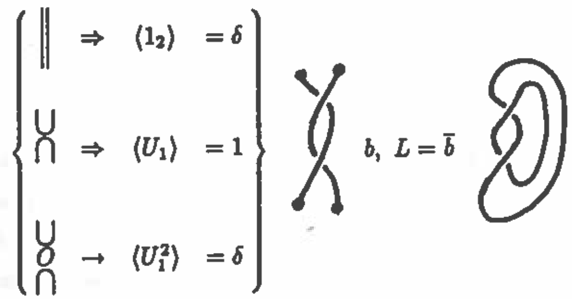
This is the braid-analog of the state expansion for the bracket.



**Example.** $b = \sigma_1^2$.

Then

$$\mathcal{U}(b) = (A + A^{-1}U_1)(A + A^{-1}U_1)$$
$$\mathcal{U}(b) = A^2 + 2U_1 + A^{-2}U_1^2$$
$$\langle b\rangle = \langle\mathcal{U}(b)\rangle = A^2\langle 1_2\rangle + 2\langle U_1\rangle + A^{-2}\langle U_1^2\rangle$$



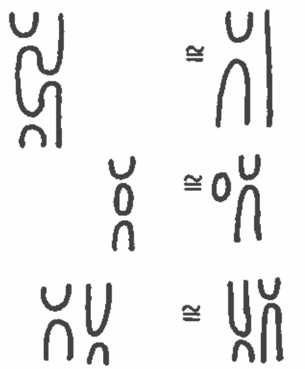

$$\therefore \langle L\rangle = A^2(-A^2 - A^{-2}) + 2 + A^{-2}(-A^2 - A^{-2})$$
$$= -A^4 - 1 + 2 - 1 - A^{-4}$$
$$\langle L\rangle = -A^4 - A^{-4}.$$

This is in accord with our previous calculation of the bracket for the simple link of two components.

The upshot of these observations is that in calculating the bracket for braids in $B_n$ it is useful to have the free additive algebra $\mathcal{A}_n$ with generators $U_1, U_2, \ldots, U_{n-1}$ and multiplicative relations coming from the interpretation of the $U_i$'s as cup-cap combinations. This algebra $\mathcal{A}_n$ will be regarded as a module over the ring $\mathbf{Z}[A, A^{-1}]$ with $\delta = -A^2 - A^{-2} \in \mathbf{Z}[A, A^{-1}]$ the designated loop value. I shall call $\mathcal{A}_n$ the **Temperley-Lieb Algebra** (see [BA1], [LK4]).

What are the multiplicative relations in $\mathcal{A}_n$? Consider the pictures in Figure 13. They illustrate the relations:

$$[\mathcal{A}]\quad \left\{\begin{array}{lcl} U_iU_{i\pm1}U_i & = & U_i \\ U_i^2 & = & \delta U_i \\ U_iU_j & = & U_jU_i, \quad \text{if } |i-j| > 1 \end{array}\right\}.$$



$U_1U_2U_1 = U_1$

$U_1^2 = \delta U_1$

$U_1U_3 = U_3U_1$

**Figure 13**

In fact, these are precisely the relations for The Temperley-Lieb algebra. Note that the Temperley-Lieb algebra and the Jones algebra are closely related. In fact, if we define $e_i = \delta^{-1}U_i$, then $e_i^2 = e_i$ and $e_ie_{i\pm1}e_i = \tau e_i$ where $\tau = \delta^{-2}$. Thus, by considering the state expansion of the bracket polynomial for braids, we recover the formal structure of the original Jones polynomial.

It is convenient to view $\mathcal{A}_n$ in a more fundamental way: Let $D_n$ denote the collection of all (topological) equivalence classes of diagrams obtained by connecting pairs of points in two parallel rows of $n$ points. The arcs connecting these points must satisfy the following conditions:

1) All arcs are drawn in the space between the two rows of points.
2) No two arcs cross one another.
3) Two elements $a, b \in D_n$ are said to be **equivalent** if they are topologically equivalent via a planar isotopy through elements of $D_n$. (That is, if there is a continuous family of embeddings of arcs - giving elements $C_t \in D_n$ $(0 \leq t \leq 1)$ with $C_0 = a$, $C_1 = b$ and $C_t$ the identity map on the subset of endpoints for each $t$, $0 \leq t \leq 1$).

Call $D_n$ the **diagram monoid on $2n$ points.**

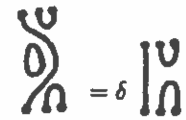
**Example.** For $n = 3$, $D_3$ has the following elements:

$1_3$ $U_1$ $U_2$ $r$ $s$



Elements of the diagram monoid $D_n$ are multiplied like braids - by attaching the output row of $a$ to the input row of $b$ - forming $ab$. Multiplying in this way, closed loops may appear in $ab$. Write $ab = \delta^k c$ where $c \in D_n$, and $k$ is the number of closed loops in the product.

For example, in $D_3$

$rs =$  $= \delta U_2.$

**Proposition 7.4.** The elements $1, U_1, U_2, \ldots, U_{n-1}$ generate $D_n$. If an element $x \in D_n$ is equivalent to two products, $P$ and $Q$, of the elements $\{U_i\}$, then $Q$ can be obtained from $P$ by a series of applications of the relations $[\mathcal{A}]$.

See [LK8] for the proof of this proposition. The point of this proposition is that it lays bare the underlying combinatorial structure of the Temperley-Lieb algebra. And, for computational purposes, the multiplication table for $D_n$ can be obtained easily with a computer program.

We can now define a mapping

$$\rho : B_n \to \mathcal{A}_n$$

by the formulas:

$$\rho(\sigma_i) = A + A^{-1}U_i$$
$$\rho(\sigma_i^{-1}) = A^{-1} + AU_i.$$

We have seen that for a braid $b$, $\langle \overline{b} \rangle = \sum_s \langle B|s \rangle \langle \overline{U}_s \rangle$ where $\rho(b) = \sum_s \langle b|s \rangle U_s$ is the explicit form of $\rho(b)$ obtained by defining $\rho(xy) = \rho(x)\rho(y)$ on products. ($s$ runs through all the different products in this expansion.) Here $\langle \overline{U}_s \rangle$ counts one less than the number of loops in $\overline{U}_s$.

Define $\mathrm{tr} : \mathcal{A}_n \to \mathbf{Z}[A, A^{-1}]$ by $\mathrm{tr}(U) = \langle \overline{U} \rangle$ for $U \in D_n$. Extend tr linearly to $\mathcal{A}_n$. This mapping - by loop counts - is a realization of Jones' trace on the von Neumann algebra $A_n$. We then have the formula: $\langle b \rangle = \mathrm{tr}(\rho(b))$.

This formalism explains directly how the bracket is related to the construction of the Jones polynomial via a trace on a representation of the braid group to the Temperley-Lieb algebra.

We need to check certain things, and some comments are in order. First of all, the trace on the von Neumann algebra $A_n$ was not originally defined diagrammatically. It was, defined in [JO7] via normal forms for elements of the Jones algebra $A_n$. Remarkably, this version of the trace matches the diagrammatic loop count. In the next section, we'll see how this trace can be construed as a modified matrix trace in a representation of the Temperley-Lieb algebra.

**Proposition 7.5.** $\rho : B_n \to A_n$, as defined above, is a representation of the Artin Braid group.

**Proof.** It is necessary to verify that $\rho(\sigma_i)\rho(\sigma_i^{-1}) = 1$, $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$ and that $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$ when $|i-j| > 1$. We shall do these in the order - first, third, second.

**First.**

$$\begin{aligned}
\rho(\sigma_i)\rho(\sigma_i^{-1}) &= (A + A^{-1}U_i)(A^{-1} + AU_i) \\
&= 1 + (A^{-2} + A^2)U_i + U_i^2 \\
&= 1 + (A^{-2} + A^2)U_i + \delta U_i \\
&= 1 + (A^{-2} + A^2)U_i + (-A^{-2} - A^2)U_i \\
&= 1
\end{aligned}$$

**Third.** Given that $|i-j| > 1$:

$$\begin{aligned}
\rho(\sigma_i\sigma_j) &= \rho(\sigma_i)\rho(\sigma_j) \\
&= (A + A^{-1}U_i)(A + A^{-1}U_j) \\
&= (A + A^{-1}U_j)(A + A^{-1}U_i), \ [U_iU_j = U_jU_i \text{ if } |i-j| > 1] \\
&= \rho(\sigma_j\sigma_i).
\end{aligned}$$



**Second.**

$$\begin{aligned}
\rho(\sigma_i\sigma_{i+1}\sigma_i) &= (A + A^{-1}U_i)(A + A^{-1}U_{i+1})(A + A^{-1}U_i) \\
&= (A^2 + U_{i+1} + U_i + A^{-2}U_iU_{i+1})(A + A^{-1}U_i) \\
&= A^3 + AU_{i+1} + AU_i + A^{-1}U_iU_{i+1} + A^{-1}U_i^2 + AU_i + A^{-1}U_{i+1}U_i \\
&\quad + A^{-3}U_iU_{i+1}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}\delta + 2A)U_i + A^{-1}(U_iU_{i+1} + U_{i+1}U_i) + A^{-3}U_i \\
&= A^3 + AU_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})U_i \\
&\quad + A^{-1}(U_iU_{i+1} + U_{i+1}U_i) \\
&= A^3 + A(U_{i+1} + U_i) + A^{-1}(U_iU_{i+1} + U_{i+1}U_i).
\end{aligned}$$

Since this expression is symmetric in $i$ and $i+1$, we conclude that

$$\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1}).$$

This completes the proof that $\rho : B_n \to A_n$ is a representation of the Artin Braid Group. //