Knots and Links in Spatial Graphs

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ABSTRACT

The main purpose of this paper is to show that any embedding of K_7 in three-dimensional euclidean space contains a knotted cycle. By a similar but simpler argument, it is also shown that any embedding of K_6 contains a pair of disjoint cycles which are homologically linked.

1. INTRODUCTION

By a *spatial embedding* of a graph Γ we mean an embedding of Γ in euclidean 3-space, which is *tame,* i.e., has a polygonal representative. Let K_n denote the complete graph on n vertices. We shall prove the following.

Theorem 1. Every spatial embedding of K_6 contains a nontrivial link.

Theorem 2. Every spatial embedding of K_7 contains a nontrivial knot.

The proofs actually yield more specific information. Precisely, every spatial embedding of K_6 contains a pair of cycles with odd linking number, and every spatial embedding of K_7 contains a Hamiltonian cycle with nonzero arf invariant.

The basic philosophy underlying both proofs is the same, although Theorem 1 is considerably easier than Theorem 2. The following is an outline of the method.

Given a spatial embedding of a graph Γ, we may suppose, after a (small) *ambient isotopy* (i.e., a continuous family of homeomorphisms h_t, $0 \leq t \leq 1$, of 3-space onto itself, such that h_0 is the identity), that the projection of Γ onto the horizontal plane is *regular;* i.e., its only singularities are double points in the interiors of edges of Γ. These may

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be indicated diagrammatically in the usual way as over/undercrossings. (See Fig. 1.) Now it is a standard fact in knot theory, not hard to prove, that any two spatial embeddings of Γ are equivalent under the equivalence relation generated by moves of the form: ambient isotopy to regular projection position followed by a change of a crossing from over to under (a crossing change). The proofs of Theorems 1 and 2 proceed by considering some (ambient isotopy) invariant of a spatial embedding of the relevant graph Γ, with the property that any embedding for which this invariant is nontrivial must necessarily satisfy the conclusion of the theorem, and then showing that (1) the invariant is unaltered by a crossing change, and (2) there exists an embedding for which the invariant is nontrivial.

Theorem 2 was proved in this way by Conway many years ago, but never published. Gordon became aware of the problem through some lectures by S. Armentrout at Princeton in April 1977, and on relating it to Conway, had an outline of the proof described to him. Expressing the details in terms of the arf invariant (as in the present paper), he subsequently reported on Conway's result in a lecture at the NSF-CBMS Conference on 3-Manifolds at VPISU, Blacksburg, in October 1977.

Theorem 1 was suggested by Ronnie Brown; it readily yields to the same general method. We are informed by Frank Quinn that this has also been done by Masayuki Yamasaki.

2. PROOFS

Knots, links, graphs, etc., will usually be unoriented. Let A_1, A_2 be disjoint graphs in euclidean 3-space, such that the projection of $A_1 \cup A_2$ is regular. Define $\omega(A_1, A_2) \in \mathbb{Z}_2$ to be the number of times (mod 2) that A_1 crosses over A_2 in the projection. (For us, A_i will actually always be an arc or a circle.)

If A_1 and A_2 are both circles, then $\omega(A_1, A_2)$ is equal to the *mod 2 linking number* of A_1 and A_2, $\text{lk}(A_1, A_2)$.

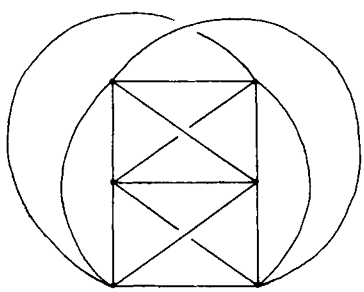

FIGURE 1.

Proof of Theorem 1. Given a spatial embedding of K_6, define $\lambda \in \mathbb{Z}_2$ by

$$\lambda = \sum \mathrm{lk}(C_1, C_2),$$

the summation being over all $10 = \frac{1}{2}\binom{6}{3}$ unordered pairs $\{C_1, C_2\}$ of disjoint cycles in K_6.

Consider what happens to λ under a crossing change. If the crossing is of an edge with itself, or of adjacent edges, then for any pair of disjoint cycles $\{C_1, C_2\}$, $\omega(C_1, C_2)$ is unchanged, and hence λ is unchanged.

If the crossing is of nonadjacent edges, A_1, A_2, say, then $\omega(C_1, C_2)$ is unchanged unless (possibly after renumbering) $A_i \subset C_i$, $i = 1,2$, in which case $\omega(C_1, C_2)$ changes by 1. But given nonadjacent edges A_1, A_2, there are exactly two such pairs $\{C_1, C_2\}$ corresponding to the choice of which of the two remaining vertices to take with A_1 to form C_1. Hence again λ is unchanged.

To complete the proof, it suffices to show that $\lambda = 1$ for some specific spatial embedding of K_6. But it is easy to check that for the embedding illustrated in Fig. 1, each pair of disjoint cycles forms a trivial link except one, which forms two unknotted circles linked once; hence $\lambda = 1$. ∎

Our proof of Theorem 2 uses the *arf invariant* $\alpha(K) \in \mathbb{Z}_2$ of a knot K. This is discussed in the Appendix. For our present purposes, we only need to know how $\alpha(K)$ is affected by a crossing change. To this end, let the knots K_+, K_-, and the 2-component link L, with components L_1, L_2, be related in that they have regular projections which are identical outside a small neighborhood where they differ as indicated in Fig. 2. We then have

Lemma 1.

$$\alpha(K_+) = \alpha(K_-) + \mathrm{lk}(L_1, L_2).$$

This is proved in the Appendix.

Proof of Theorem 2. Given a spatial embedding of K_7, define $\sigma \in \mathbb{Z}_2$ by

$$\sigma = \sum \alpha(C),$$

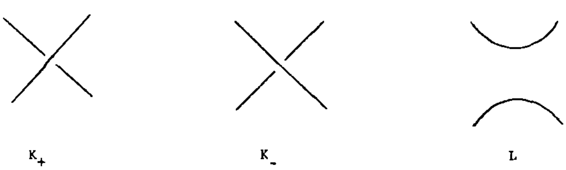

FIGURE 2.

the summation being over all $360 = \frac{1}{2}6!$ Hamiltonian cycles C in K_7. We shall show, using Lemma 1, that σ is invariant under crossing changes.

The crossings are of three kinds: of an edge with itself, of adjacent edges, and of distinct nonadjacent edges. Note, however, that we need never consider crossings of an edge with itself, as a change in such a crossing can always be replaced by five changes of crossings of distinct edges. (The process is indicated schematically in Fig. 3.)

Also, if we want to change a crossing of adjacent edges A, B, we may first contract A, say, by moving its vertices along itself toward the crossing point in question, dragging the rest of the graph along, and in the same way move the vertex of B which does not belong to A toward the crossing point. Thus we may assume that the projection of K_7 near our crossing point is as shown in Fig. 4(a), possibly with the crossing reversed.

Similarly, for a change of crossing of two distinct nonadjacent edges A, B, by contracting each edge toward the crossing point we may assume that the projection near this crossing point is exactly as in Fig. 5(a) (possibly with the crossing reversed).

Hence it will suffice to show that σ is invariant under these two kinds of *special* crossing changes. (This geometrical simplification is not strictly necessary, but it does shorten somewhat the counting argument which follows.)

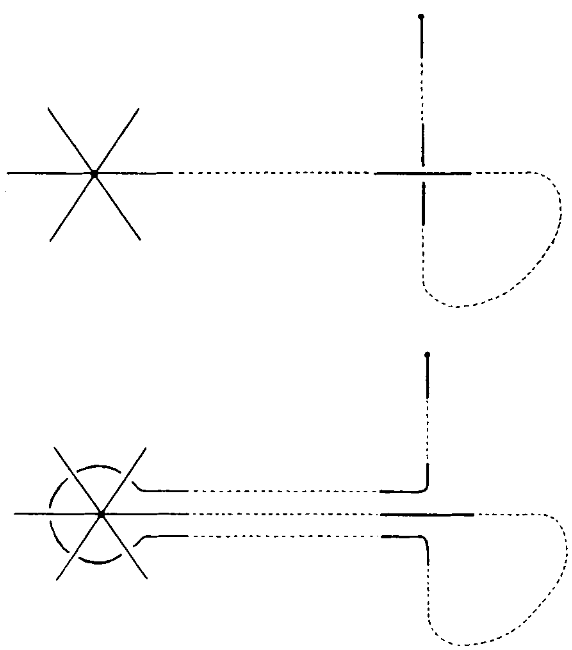

FIGURE 3.

FIGURE 4.

Consider, then, a special change of crossing of (distinct) edges A, B. Certainly $\alpha(C)$ is unchanged if C does not contain A and B, so let C be a Hamiltonian cycle which does. Let $\varepsilon(C) \in \mathbb{Z}_2$ denote the change in $\alpha(C)$ induced by the crossing change. By Lemma 1, $\varepsilon(C) = \mathrm{lk}(L_1, L_2)$, where $L = L_1 \cup L_2$ is the link determined by C and the crossing change as described in that lemma. We consider the two kinds of special crossing changes separately.

I. Suppose A and B are adjacent. Then $L = L_1 \cup L_2$ is as indicated in Fig. 4(b). Note that L_1 is independent of C. We have

$$\varepsilon(C) = \mathrm{lk}(L_1, L_2) = \sum \omega(L_1, E),$$

the summation being over all edges $E \subset C$, $E \neq A, B$. The change in σ is $\Sigma\ \varepsilon(C)$, summed over all Hamiltonian cycles C containing A and B.

Now for any edge E in K_7, $E \neq A, B$, the number of Hamiltonian cycles containing A, B, and E is

0, if E, A, B have a common vertex;

3!, if E is adjacent to A or B (but not both);

$2 \times 3!$, otherwise.

Hence for any edge $E \neq A, B$ in K_7, $\omega(L_1, E)$ appears an even number of times in $\Sigma\ \varepsilon(C)$. Therefore $\Sigma\ \varepsilon(C) = 0$, showing that σ is unchanged.

II. Let A, B be distinct nonadjacent edges. Here the link $L = L_1 \cup L_2$ is as indicated in Fig. 5(b). We have

$$\varepsilon(C) = \mathrm{lk}(L_1, L_2) = \sum \omega(E_1, E_2),$$

FIGURE 5.

summed over all pairs of edges $\{E_1, E_2\}$ of C such that $E_i \subset L_i$, $i = 1, 2$.

But for any pair $\{E_1, E_2\}$ of edges of K_7, neither of which is A or B, it is easy to verify that if $\nu(E_1, E_2)$ denotes the number of Hamiltonian cycles C containing A and B such that (possibly after renumbering) $E_i \subset L_i$, $i = 1, 2$, then $\nu(E_1, E_2)$ is always even. In fact, if we label the vertices of K_7 as 1, 2, ..., 7, and use (ij) to denote the edge with vertices i, j, then we may take $A = (12)$, $B = (34)$, and assume that the vertices 2, 3 and 1, 4 are paired in forming L (see Fig. 6). Then, up to symmetry, the only cases in which $\nu(E_1, E_2)$ is nonzero are with E_1, E_2 equal to

(i) (23), (45); (ii) (23), (56);

(iii) (27), (45); (iv) (27), (56);

the corresponding Hamiltonian cycles being

(i) (1234567), (1234576);

(ii) (1234567), (1234657), (1234756), (1234765);

(iii) (1273456), (1276345);

(iv) (1273456), (1273465).

It follows that in $\Sigma\, \varepsilon(C)$, summed over all Hamiltonian cycles C containing A and B, each term $\omega(E_1, E_2)$ appears an even number of times. Hence again σ is unchanged.

Finally, it is a routine exercise to verify that the embedding of K_7 shown in Fig. 7 has all Hamiltonian cycles unknotted except one, which is a trefoil knot. Since the arf invariant of the trefoil is 1, this embedding has $\sigma = 1$, and the proof is complete. ∎

Appendix

In this appendix we briefly describe two approaches to the arf invariant of a knot.

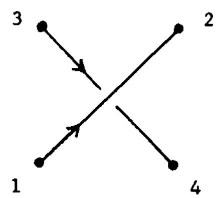

FIGURE 6.

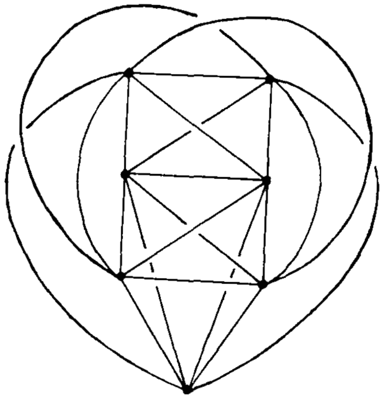

FIGURE 7.

I. Let K be a knot in euclidean 3-space E^3, and let F be an orientable surface spanning K. Since F is two sided, there corresponds to each 1-cycle C on F a 1-cycle $\hat{C}$ in $E^3 \setminus F$ obtained by pushing C off F in some fixed normal direction. Define

$$\varphi\colon H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$$

by

$$\varphi([C]) = \operatorname{lk}(\hat{C}, C).$$

This is a quadratic function whose associated bilinear form $\beta(x, y) = \varphi(x + y) + \varphi(x) + \varphi(y)$ is readily identified with the mod 2 homology intersection form on F. Since the latter is nonsingular, the *arf invariant* of φ is defined, and is given by

$$\alpha(\varphi) = \sum_{i=1}^{n} \varphi(e_i)\varphi(f_i),$$

where $e_1, f_1, \ldots, e_n, f_n$ is any symplectic basis for β [1]. It can be shown that $\alpha(\varphi)$ is independent of the particular choice of F, enabling one to define the *arf invariant of* K by $\alpha(K) = \alpha(\varphi)$ [7].

To prove Lemma 1, let F_+ be an orientable spanning surface for K_+. It is easy to see that near the indicated crossing point, we may assume that F_+ is as illustrated in Fig. 8. There are then defined corresponding orientable spanning surfaces F_-, F_0 for K_-, L, respectively, as illustrated, such that $F_\pm$ is obtained by joining the boundary components of F_0 by

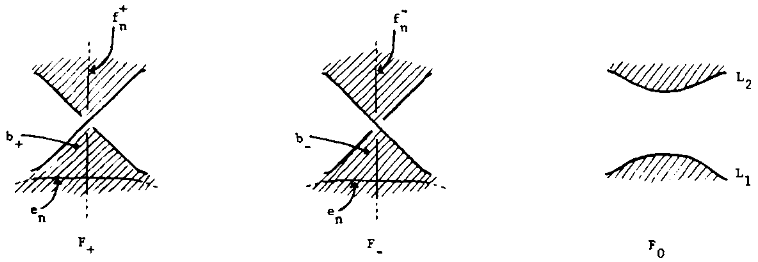

FIGURE 8.

a band $b_\pm$. By adding a tube to it if necessary, we may assume that F_0 is connected.

Let $e_1, f_1, \ldots, e_{n-1}, f_{n-1}$ be a symplectic basis for $H_1(F_0; \mathbb{Z}_2)$. This can be extended to a symplectic basis for $H_1(F_\pm; \mathbb{Z}_2)$ by adjoining $e_n, f_n^\pm$, where e_n is represented by L_1, say, and $f_n^\pm$ is represented by the union of the core of $b_\pm$ with a suitable path in F_0 joining the ends of this core. (See Fig. 8.) Note that since the bands $b_\pm$ differ by a single twist, $\varphi(f_n^+) - \varphi(f_n^-) = 1$. Hence

$$\begin{aligned}\alpha(K_+) - \alpha(K_-) &= \varphi(e_n)[\varphi(f_n^+) - \varphi(f_n^-)] \\ &= \varphi(e_n) \\ &= \operatorname{lk}(\hat{L}_1, L_1) \\ &= \operatorname{lk}(\hat{L}_1, L_2)\end{aligned}$$

(since L_1 and L_2 are homologous, by F_0, in the complement of $\hat{L}_1$)

$$= \operatorname{lk}(L_1, L_2)$$

(since L_1 and $\hat{L}_1$ are homologous in the complement of L_2).

II. Let $\nabla_K(z) = \sum_{i=0}^{\infty} a_i(K)z^i$ denote the Conway polynomial of the oriented knot or link K [2, 3]. Define $\alpha(K) \in \mathbb{Z}_2$ to be the mod 2 reduction of $a_2(K)$. If K is a knot, then it turns out that $\alpha(K)$ coincides with the arf invariant of K as defined previously. {This follows easily (see [3]) from the fact that, using the first definition of $\alpha(K)$, $\alpha(K) = 0$ or 1 according as $\Delta(-1) \equiv \pm 1$ or ± 3 (mod 8), where Δ is the Alexander polynomial of K [4, 6].}

Referring to Fig. 2, recall [2, 3] that, with suitable string orientations, we have the identity

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_L.$$

(We no longer insist that $K_\pm$ have only one component.) In particular,

$$a_0(K_+) - a_0(K_-) = 0, \tag{i}$$

$$a_1(K_+) - a_1(K_-) = a_0(L), \tag{ii}$$

$$a_2(K_+) - a_2(K_-) = a_1(L). \tag{iii}$$

It follows readily from (i) by an inductive argument that $a_0(K) = 1$ or 0 according as K has one or more than one component, and in the same way (ii) then implies that $a_1(K)$ is equal to the (integral) linking number of the components of K if K has two components, and 0 otherwise. Lemma 1 now follows immediately from (iii).

Finally, we remark that a four-dimensional proof of Lemma 1 is given in [5].

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