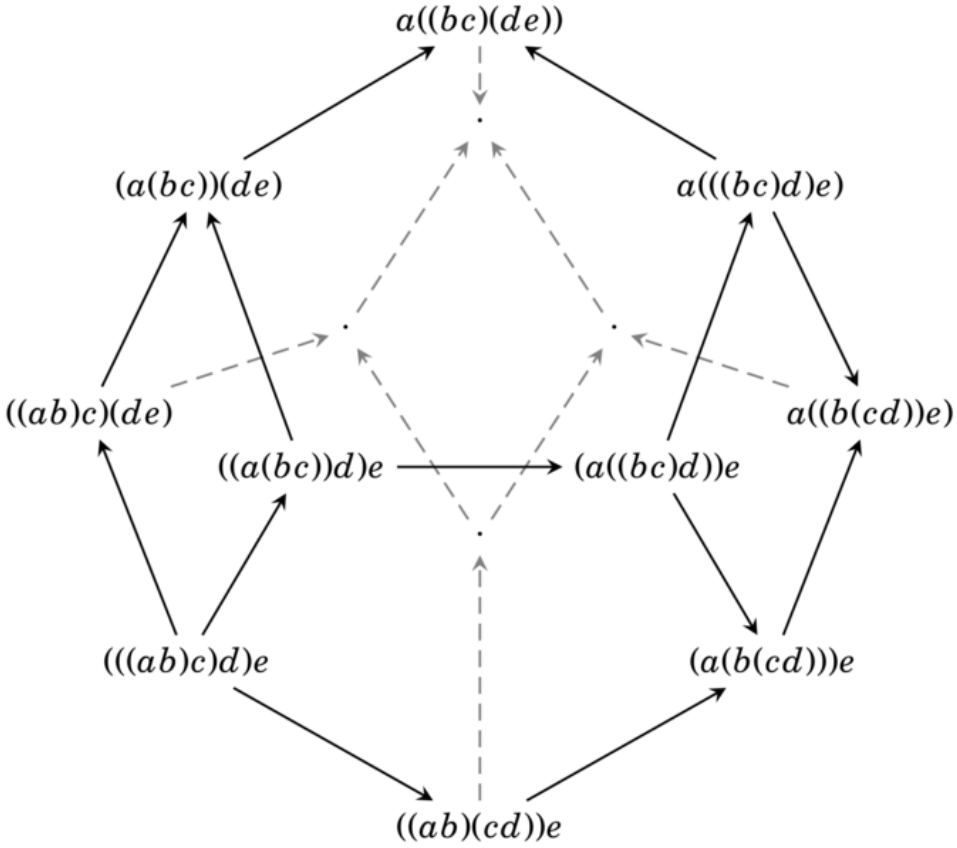
Discrete Structures

Problems and Supplemental Material



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This document contains in-class problems and supplemental readings for Math 113. The version hosted on the course website will have solutions appended as we cover problems. If you find any typos or have comments/suggestions, please contact me at ormsbyk@reed.edu.

Contents

CHAPTER 1

Combinatorics

1. Day 1

QUESTION 1.1 (Non-attacking rooks). Rooks are chess pieces which move vertically and horizontally. We say that two rooks are attacking each other if they are in the same rank (*i.e.* row) or file (*i.e.* column). Is it possible to place 8 rooks on a standard 8×8 chessboard so that no two rooks are attacking each other? In how many different ways can non-attacking rooks be placed on the board? What if the chessboard is $n \times n$ and you have n rooks?

FIGURE 1. Pacifist rooks on a 2×2 chessboard.

QUESTION 1.2 (Monotonic paths). A path on a square grid is called *monotonic* if it proceeds only by single steps right or up. On a 4×4 (or $n \times k$) grid, how many distinct monotonic paths go from the bottom left corner to the top right corner? What does this have to do with Figure 2? (To make indexing easier, you may want to assume that your grid has $(0, 0)$ as its bottom left corner and (n, k) as its top right corner.)

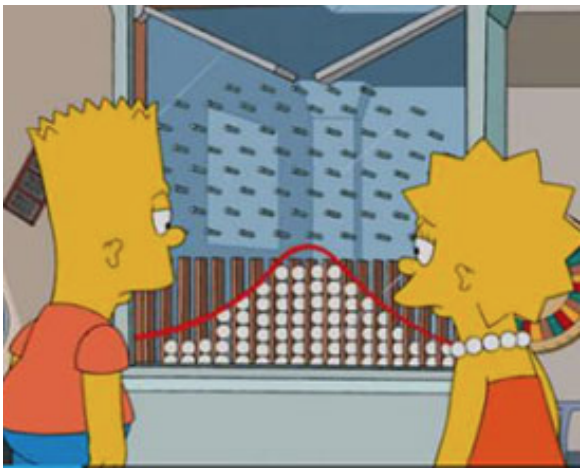


FIGURE 2. Bart and Lisa experience the Galton board

2. Day 2

This course employs two main counting principles: the *additive counting principle* (ACP) and the *multiplicative counting principle* (MCP).

Let S be a *finite set*, *i.e.*, a finite collection of objects. A *partition* of S is a way to divy S up into pieces that do not overlap; more precisely, we write

$$S = S_1 \amalg S_2 \amalg \cdots S_m$$

and call $\{S_1, \ldots, S_m\}$ a partition of S if $S_1, S_2, \ldots, S_m$ are sub-collections of S such that every object in S is in *exactly one* of the S_i. In this situation, if we want to count S, then we can count each of the S_i and then add up the totals. This is exactly what the ACP says:

THEOREM 2.1 (Additive Counting Principle). *If $\{S_1, S_2, \ldots, S_m\}$ is a partition of S, then*

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

Here the bar notation $|S|$ indicates the *cardinality* of S, *i.e.*, the number of objects in S. We trust that the reader finds this principle sufficiently obvious and will not provide a formal proof. There will be many situations in which our counts will break into disjoint pieces or cases, and this is when we will employ the ACP.

The multiplicative counting principle imposes a uniformity condition on the partition and deduces a simpler formula.

THEOREM 2.2 (Multiplicative Counting Principle – Version 1). *If $\{S_1, S_2, \ldots, S_m\}$ is a partition of S and each S_i has the same cardinality n, then*

$$|S| = mn.$$

PROOF. By hypothesis, $|S_1| = |S_2| = \cdots = |S_m| = n$, and by the ACP,

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

Substituting, we get

$$|S| = \underbrace{n + n + \cdots + n}_{m \text{ times}} = mn,$$

as desired. □

We will frequently employ a variant of the MCP in which we count choices. Suppose that we are making two-person teams, where the first team member has an early birthday (between January and June), and the second team member has a late birthday (between July and December). Let S be the set of all two-person teams, and enumerate the early birthday individuals $e_1, e_2, \ldots, e_m$. For $1 \leq i \leq m$, let S_i be the set of teams with early birthday member e_i. How large is S_i? If the late birthday individuals are $l_1, l_2, \ldots, l_n$, then e_i can be paired with any of these n individuals. Thus $|S_i| = n$ for all i, and $\{S_1, \ldots, S_m\}$ is a partition of S. We conclude by the MCP that there are mn such teams.

But we can rephrase this count in the following way: we had m choices for how to pick the first team member, and then n choices for how to pick the second. Thus there are mn many teams. This is our second version of the MCP.

THEOREM 2.3 (Multiplicative Counting Principle – Version 2). *If we can enumerate the elements of S* (i.e., *count them without repetition) by first making m choices and then making n choices, then $|S| = mn$. More generally, if we can enumerate S by making m_1 choices, then making m_2 choices,* etc., *until finally making m_k choices, then*

$$|S| = m_1 m_2 \cdots m_k.$$

The proof is by iterative application of the two-choice case, which we have already justfied. We will provide a formal justification after we have studied mathematical induction, but you are free to use Theorem 2.3 now.

In many cases, a natural counting scheme will overcount by a consistent factor. You saw this in the handshake problem: there are n ways to choose a first person, and then $n-1$ ways to choose the second person, but the pair (A, B) and (B, A) constitute the same handshake, so the MCP count $n(n-1)$ overcounts by a factor of 2; the total number of handshakes possible between n people is $n(n-1)/2$.

PROPOSITION 2.4 (Overcounting Principle). If a method of counting a finite set S results in a total count of N but counts each element of S a total of n times, then

$$|S| = \frac{N}{n}.$$

We will revisit and formally justify this intuitive principle after we study equivalence relations.

QUESTION 2.5. In how many distinct ways can the letters in the word MISSISSIPPI be arranged?

3. Day 3

PROBLEM 3.1. Is it always the case that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$? Draw a picture to support your assertion and then prove it.

Cartesian product. There is another operation on sets called the *Cartesian product*. For sets A and B, their Cartesian product is the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\},$$

the collection of ordered pairs where the first element is in A and the second is in B.

QUESTION 3.2. Big Brothers Big Sisters of Portland has a collection A of 30 adult volunteers and group C of 50 children in need of an adult partner. What is a set which describes the possible adult-child pairings? How many adult-child pairings exist?

PROBLEM 3.3. Find a general formula for $|A \times B|$ in terms of $|A|$ and $|B|$.

Functions. Functions are ways of relating one set to another. Thus to each element a of a set A, a function assigns exactly one element $b \in B$. If the function's name is f, then we write $b = f(a)$.

The set A is called the *domain* of f and B is its *codomain* (aka *range*). This can all be compactly expressed via the notation $f\colon A \to B$.

Each function $f\colon A \to B$ has an associated *graph* $G_f = \{(a, f(a)) \mid a \in A\} \subseteq A \times B$. A generic subset $G \subseteq A \times B$ is the graph of a function if and only if for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in G$. In set theory (which aims to express every mathematical concept in terms of sets), a function is actually defined to be such a special subset of $A \times B$. It's good to be aware of this formalism, but more useful in everyday mathematical practice to think of functions as assignments.

PROBLEM 3.4. Which of the following subsets of $\{1, 2, 3\} \times \{a, b, c, d\}$ are functions?

(a) $\{(1, a), (2, b), (3, d)\}$
(b) $\{(2, d), (3, c)\}$
(c) $\{(1, b), (2, c), (3, a), (2, d)\}$
(d) $\{(1, a), (2, a), (3, a)\}$

4. Functions

4.1. Functions as assignments. On Day 3, we defined a function $f\colon A \to B$ (with domain the set A and codomain the set B) to be a subset $f \subseteq A \times B$ such that for every $a \in A$ there is a unique pair $(a, b) \in f$. This is the *graph* interpretation of functions: think of A as the "horizontal axis" and B as the "vertical axis." (If A and B are (subsets of) $\mathbb{R}$, you can literally do this!) The function condition is then the "vertical line test" — each "vertical line" through some $a \in A$ hits exactly one graphed point (a, b).

It is typical to think of functions as *assignments* rather than as particular subsets of a Cartesian product. When $(a, b) \in f\colon A \to B$, we say that $b = f(a)$ and think of f "sending" a to b. The function condition then says that each $a \in A$ gets sent to precisely one $b \in B$.[1]

EXAMPLE 4.1. Consider the set $f = \{(1, 3), (2, 3), (3, 4)\} \subseteq \{1, 2, 3\} \times \{1, 2, 3, 4\}$. This is a function for which $f(1) = 3$, $f(2) = 3$, and $f(3) = 4$.

NOTATION 4.2. We will sometimes write $f\colon a \mapsto b$ when $f(a) = b$ and read this statement as "f maps a to b." It is important that "$\mapsto$" is not the same as "$\to$": $f\colon A \to B$ tells us that f is a function with domain A and codomain B, while $f\colon a \mapsto b$ says that $f(a) = b$. For the function from Example 4.1, we could write $f\colon 1 \mapsto 3,\ 2 \mapsto 3,\ 3 \mapsto 4$.

EXAMPLE 4.3. In calculus, you may have considered a function $\mathbb{R} \to \mathbb{R}$ given by a formula such as $f(x) = x^3 + \sin x$. This is still a perfectly reasonably function because each $x \in \mathbb{R}$ is sent to one $f(x) \in \mathbb{R}$ (namely, $x^3 + \sin x$). As a graph, this function is $\{(x, x^3 + \sin x) \mid x \in \mathbb{R}\}$.

EXAMPLE 4.4. Not all functions have reasonable formulæ. For instance, there is a function $g\colon \mathbb{R} \to \mathbb{R}$ which takes x to x if the first nonzero digit of x is 1 and otherwise takes x to 0. Weird, but still a function.[2]

EXAMPLE 4.5. Here's an interesting way to use a function: Given a set X and subset $A \subseteq X$, let's build a function which specifies the points of A. We define the *indicator function* for A to be $\chi_A\colon X \to \{0, 1\}$ given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

A couple of comments: first, χ is the Greek letter "chi." Second, the formula above is an example of a *piecewise definition*: we partition the domain into disjoint subsets whose union is all of X (in this case, A and $X \setminus A$), and then give a formula or rule describing what the function does to elements in each subset.

Note that we can reconstruct A from χ_A as all $x \in X$ such that $\chi_A(x) = 1$, *i.e.*,

$$A = \{x \in X \mid \chi_A(x) = 1\}.$$

Keep this example in mind when you read about the enumeration of subsets via binary sequences!

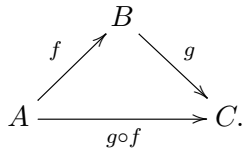
4.2. Composition. Let's now explore how functions interact with each other via composition.

DEFINITION 4.6. Suppose $f\colon A \to B$ and $g\colon B \to C$ are functions and the codomain of f equals the domain of g. Then we define the *composite* of g with f to be the function $g \circ f\colon A \to C$ by the equation $(g \circ f)(a) = g(f(a))$.

[1] Note that for a given $b \in B$, more than one a can go to b. The point here is that (1) $f(a)$ takes some value in B, and (2) it only takes one, instead of multiple, values in B.

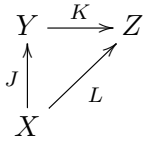
[2] Worse yet, "most" functions between infinite sets are not describable by any written rule whatsoever, but we will not pursue this perversity further.

The composite $g \circ f$ "does f first" and then "does g." We can express this graphically with a picture called a commutative diagram:



Here the arrows go from domain to codomain and are labelled by the corresponding function. If we start with $a \in A$, then the arrow labelled f takes a to $f(a)$. Continuing this path, the arrow labelled g takes $f(a)$ to $g(f(a))$. Meanwhile, the arrow labelled $g \circ f$ takes a to $g(f(a))$ by definition. Since both paths do the same thing to every $a \in A$, we say that it "commutes."

The exact shape of a commutative diagram doesn't matter. If someone told us that the diagram



commutes, we would know that $K(J(x)) = L(x)$ for each $x \in X$; in other words, $L = K \circ J$ when that diagram commutes.

We can compose more than two functions as well, as long as domains and codomains match up properly. For instance, $h \circ g \circ f \colon A \to D$ makes sense as long as $f \colon A \to B$, $g \colon B \to C$, and $h \colon C \to D$ for some sets A, B, C, and D; we have $(h \circ g \circ f)(a) = h(g(f(a)))$. We leave it as an exercise to the reader to (a) check that $h \circ g \circ f = h \circ (g \circ f) = (h \circ g) \circ f$, and (b) draw a commutative diagram describing this triple composite. Property (a) has a name: composition is *associative*.

Every set A supports a special function $\mathrm{id}_A \colon A \to A$, called the *identity function* on A, which interacts in a special way with composition. This function simply takes a to a for each $a \in A$, *i.e.*, $\mathrm{id}_A \colon a \mapsto a$ or $\mathrm{id}_A(a) = a$. If $f \colon A \to B$ is a function, let's consider the composite $f \circ \mathrm{id}_A$. Well, $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a)$ for every $a \in A$, so $f \circ \mathrm{id}_A = f$. Similarly, $\mathrm{id}_B \circ f = f$. (Note that we had to change id_A to id_B so that domains and codomains would match up!) We see then that composition with the identity function *does nothing* to the other function. This distinguishes identity functions amongst all functions with the same domain and codomain.

4.3. Special types of functions. We now explore functions with special properties, namely *injections*, *surjections*, and *bijections*.

Injections. An injection is a function which does not hit the same value twice. We formalize this idea in the following definition.

DEFINITION 4.7. A function $f \colon A \to B$ is *injective* (or is an *injection*) if $f(x) = f(y)$ (for $x, y \in A$) if and only if $x = y$.

Meditate on this definition for a while if it seems funny. The point is that f does not duplicate values in the codomain, so an equality between values ($f(x) = f(y)$) is only possible when $x = y$.

Let's briefly return to our graph interpretation of functions. An injection hits each value in the codomain at most once. This is also referred to as the *horizontal line test*: when we draw a horizontal line through any $b \in B$, we hit at most one point of the form (a, b) in the graph.

You may have learned in middle school that functions passing the horizontal line test have inverses. This fact remains true in the current context, although we must be careful with the domain of our inverse function, requiring the following definition.

DEFINITION 4.8. The *image* of a function $f\colon A \to B$ is the set

$$\operatorname{im}(f) = \{b \in B \mid \text{there exists } a \in A \text{ such that } f(a) = b\}.$$

In other words, the image of f consists of all the elements of B that are "hit" by the function. For instance, the image of the function $f\colon \{1,2,3\} \to \{1,2,3,4\}$ from Example 4.1 is $\{3,4\}$. The image of the function from Example 4.4 is

$$\{x \in \mathbb{R} \mid \text{the first nonzero digit of } x \text{ is } 1\} \cup \{0\}.$$

When a function $f\colon A \to B$ is injective, it has an *inverse* function $f^{-1}\colon \operatorname{im}(f) \to A$; this is the unique function satisfying the equalities $f(f^{-1}(b)) = b$ for each $b \in \operatorname{im}(f)$ and $f^{-1}(f(a)) = a$ for each $a \in A$. It is tempting then to write that $f \circ f^{-1} = \mathrm{id}_{\operatorname{im}(f)}$ and $f^{-1} \circ f = \mathrm{id}_A$, but we should recognize that there is a slight mismatch between domains and codomains. If we replace $f\colon A \to B$ with $\tilde{f}\colon A \to \operatorname{im}(f)$ taking the same values ($\tilde{f}(a) = f(a)$ for all $a \in A$), then its completely legitimate to write $\tilde{f} \circ f^{-1} = \mathrm{id}_{\operatorname{im}(f)}$ and $f^{-1} \circ \tilde{f} = \mathrm{id}_A$.

Surjections. Given the terminology we've already introduced, surjections are easy to define.

DEFINITION 4.9. A function $f\colon A \to B$ is *surjective* (or is a *surjection*) if $\operatorname{im}(f) = B$.

In other words, surjections hit everything in their codomain. Of course, when we define a function, we have some choice regarding the codomain. For instance, we could consider the assignment on real numbers $x \mapsto x^2$ to have codomain $\mathbb{R}$ or codomain $[0,\infty) = \{x \in \mathbb{R} \mid x \geq 0\}$. In the first instance, the function is not surjective, but in the latter case it is (because every nonnegative real number has a square root [in fact, two square roots]).

EXAMPLE 4.10. Suppose $A \subsetneq X$ is a nonempty proper subset of X. Then the indicator function $\chi_A\colon X \to \{0,1\}$ is surjective. (Why? What if $A = \varnothing$ or X?)

Bijections. Finally, we come to bijections, also called one-to-one correspondences.

DEFINITION 4.11. A function is *bijective* (or is a *bijection*) if it is both injective and surjective.

Suppose $f\colon A \to B$ is bijective. Then it is injective with $\operatorname{im}(f) = B$, so it has an inverse function of the form $f^{-1}\colon B \to A$ satisfying $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. (We don't need to replace f with $\tilde{f}$ because $\operatorname{im}(f)$ is all of B.) In fact, a function has such an inverse if and only if it is bijective.

THEOREM 4.12. *A function $f\colon A \to B$ is bijective if and only if there exists a function $g\colon B \to A$ (called a [two-sided] inverse of f) such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.*

PROOF. We have already seen that if f is bijective, then such a g exists. Suppose now that $f\colon A \to B$ is a function and there exists $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. We need aim to show that f is bijective, and will first show that it is injective. Suppose that there are $x, y \in A$ such that $f(x) = f(y)$. Applying g to this equality, we get $g(f(x)) = g(f(y))$, and since $g \circ f = \mathrm{id}_A$, this becomes $x = y$. Hence f is injective.

We now show that f is surjective. Given $b \in B$, let $a = g(b)$. Then $f(a) = f(g(b)) = b$, so f is surjective. Since f is injective and surjective, it is in fact a bijection, as desired. □

Bijections are incredibly useful in combinatorics. Every combinatorial problem can be reframed as trying to determine the cardinality of a set. The following theorem tells us that bijections preserve cardinality, so a good way to "count" is to produce a bijection between the set we would like to count, and a set with a known number of elements.

THEOREM 4.13. *There exists a bijection $f\colon A \to B$ between finite sets A and B if and only if $|A| = |B|$.*

PROOF. Suppose that $|A| = n = |B|$. By counting the n elements of A and B, we produce bijections $a\colon \{1, 2, \ldots, n\} \to A$ and $b\colon \{1, 2, \ldots, n\} \to B$. You should check that $f = b \circ a^{-1}$ is a bijection $A \to B$.

Now suppose that A is finite of cardinality n and there exists a bijection $f\colon A \to B$. Counting A again produces a bijection $a\colon \{1, 2, \ldots, n\} \to A$. Convince yourself that $f \circ a\colon \{1, 2, \ldots, n\} \to B$ counts B, so $|B| = n$ as well. □

5. Day 4

The *floor* function $\lfloor\ \rfloor\colon \mathbb{R} \to \mathbb{R}$ sends $x \in \mathbb{R}$ to the greatest integer less than or equal to x. For instance, $\lfloor 4.5 \rfloor = 4$, $\lfloor 17 \rfloor = 17$, and $\lfloor -\pi \rfloor = -4$.

PROBLEM 5.1. Draw a graph of $\lfloor\ \rfloor$ and check that it is a function. What is the image of the floor function? Is it injective or surjective?

PROBLEM 5.2. Define $f\colon \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{-1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that f is a bijection.

PROBLEM 5.3. Suppose A and B are finite sets and $f\colon A \to B$ is injective. What can we say about $|A|$ and $|B|$? What if f is surjective?

PROBLEM 5.4. Let $F(A, B)$ denote the set of functions with domain A and codomain B. If $|A|, |B| < \infty$, what is $|F(A, B)|$? (In other words, how many functions are there with domain A and codomain B?)

Suppose A and B are sets and $f\colon A \to B$ is a function. If $A' \subseteq A$, then the *image* of A' in B is defined as

$$f(A') := \{f(a) \mid a \in A'\}.$$

Note that $f(A) = \mathrm{im}(f)$. If $B' \subseteq B$, then the *preimage* of B' in A is defined as

$$f^{-1}(B') := \{a \in A \mid f(a) \in B'\}.$$

In other words, $f^{-1}(B')$ consists of everything in A pushed into B' by f.

PROBLEM 5.5. Determine $f(\varnothing)$ and $f^{-1}(\varnothing)$. More generally, when is $f^{-1}(B') = \varnothing$?

PROBLEM 5.6. For $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$, and $f\colon A \to B$, prove that

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2),$$
$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2),$$
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2), \text{ and}$$
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$$

Find an example to show that equality does not necessarily hold in the second line.

6. Day 5

PROBLEM 6.1. If $a_0, a_1, a_2 \ldots, a_k \in \{0, 1\}$, we write $(a_k a_{k-1} \ldots a_2 a_1 a_0)_2$ for the integer represented by this string in base 2; in other words,

$$(a_k a_{k-1} \ldots a_2 a_1 a_0)_2 = a_k 2^k + a_{k-1} 2^{k-1} + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0.$$

(a) How do you express $2 \cdot (a_k a_{k-1} \ldots a_2 a_1 a_0)_2$ in binary?

(b) Find a closed formula for the n-th term in the sequence 1_2, 11_2, 111_2, 1111_2, ...

PROBLEM 6.2. Suppose A is a nonempty finite set containing n elements and that a is a particular element of A. How many subsets of A contain a? (Try to solve this problem both with a direct count, and also by producing a bijection between $\{B \subseteq A \mid a \in B\}$ and a set which you've already counted.)

PROBLEM 6.3. Determine the number of ordered pairs (A, B) where

$$A \subseteq B \subseteq \{1, 2, \ldots, n\}.$$

PROBLEM 6.4. In what number system can you easily enumerate the pairs in Problem 6.3? Use this number system to enumerate such pairs when $n = 3$.

PROBLEM 6.5. Generalize the above two problems to finite "chains of subsets" $(A_1, A_2, \ldots, A_m)$ where

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq \{1, 2, \ldots, n\}.$$

7. Day 6

For $n \in \mathbb{N}$, let $\underline{n} = \{1, 2, \ldots, n\}$. In particular $\underline{1} = \{1\}$, $\underline{2} = \{1, 2\}$, $\underline{3} = \{1, 2, 3\}$, etc. Note that $\underline{0} = \varnothing$ by convention.

PROBLEM 7.1. There are k^n length n strings where each entry in the string comes from a set with k elements. Earlier, you proved that there are k^n functions with domain $\underline{n}$ and codomain $\underline{k}$. Is this a coincidence? Explain.

We take the viewpoint that a permutation is a bijection from a set to itself. This can also be though of as a reordering of the set. If $\pi : \underline{n} \to \underline{n}$ is a bijection, it reorders $\underline{n}$ from $1, 2, \ldots, n$ to $\pi(1), \pi(2), \ldots, \pi(n)$. This also gives us the SAT-style analogy

$$\text{string} : \text{function} :: \text{reordering} : \text{permutation}.$$

In particular, we may view permutations of $\underline{n}$ as length n strings with entries in $\underline{n}$ in which no 'letters' are repeated.

PROBLEM 7.2. Why does this prove that $n! \leq n^n$? What do you think $n!/n^n$ approaches as n goes to ∞?

Define the *sign* of a permutation $\pi : \underline{n} \to \underline{n}$ by the formula

$$\operatorname{sgn}(\pi) = \prod_{1 \leq i < j \leq n} \frac{\pi(j) - \pi(i)}{j - i}.$$

Here $\prod$ stands for product, and we are taking the product of the factors $\frac{\pi(j)-\pi(i)}{j-i}$ as i and j range over all pairs of integers (i, j) with $1 \leq i < j \leq n$.

PROBLEM 7.3 (Challenge). Write out the formula for $\operatorname{sgn}(\pi)$ when $n = 3$. Why is it the case that $\operatorname{sgn}(\pi) = \pm 1$ in this case? Show that $\operatorname{sgn}(\pi) \in \{\pm 1\}$ for all n.

8. Equivalence relations

Consider the problem of putting King Arthur and his twelve knights in a line. Thirteen different people can take the first spot in line, twelve can take the second, *etc.*, until there is only one person who can take the final spot. We deduce that there are

$$13 \cdot 12 \cdot 11 \cdots 2 \cdot 1 = 13!$$

ways for the heroes of Camelot to queue up.

Note, though, that Arthur and his knights are famous enough that they rarely have to wait in line. With the extra leisure time this affords, they like to sit at the Round Table. Since the table is round, we consider seatings to be "the same" or "equivalent" if one can be rotated to produce the other. (Rotation by 0° counts, so any given seating is equivalent to itself.)

With this notion of rotational equivalence in hand, we can break up the queuings of the first paragraph into "equivalence classes" of seatings that can be rotated into each other. Since each such equivalence class consists of 13 lineups, there are a total of

$$13!/13 = 12!$$

seatings that cannot be rotated into each other.

Our task in these notes is to formalize the above ideas and see how they fit into combinatorics.

8.1. Definitions and examples.

DEFINITION 8.1. A *relation* R on a set A is a subset of $A \times A$. We write aRb when $(a, b) \in R$.

The idea here is to think of a being Related (somehow) to b when aRb, *i.e.*, when $(a, b) \in R$. It is also common to use a special symbol such as $\sim$, $\simeq$, $\cong$, or $\equiv$ to denote a relation. The particular symbols just mentioned are more common when the relation is in fact an equivalence relation, which we presently define.

DEFINITION 8.2. A relation $\sim$ on A is an *equivalence relation* if it is

(a) *reflexive*: for all $a \in A$, $a \sim a$,
(b) *symmetric*: for $a, b \in A$, if $a \sim b$, then $b \sim a$, and
(c) *transitive*: for $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Let S denote the set of students in a class. We can define an equivalence relation $\cong$ on S by declaring that $s \cong t$ if and only if s and t have the same birthday. Let's check that it forms an equivalence relation. Clearly for each $s \in S$, s has the same birthday as s, so $s \cong s$. If s has the same birthday as t, then t has the same birthday as s, so $s \cong t$ implies that $t \cong s$. Finally, if s has the same birthday as t and t has the same birthday as u, then s has the same birthday as u, so the relation is transitive. We conclude that $\cong$ is an equivalence relation on S.

Now consider the King Arthur problem again. To make life easier, let's number the Camelotians 1, 2, 3, ..., 13. Let Q denote the set of queues of $1, 2, \ldots, 13$, *i.e.*, the set of permutations of $\underline{13} = \{1, 2, \ldots, 13\}$. Two queues create the same seating if we can cyclically reorder (rotate the table) from one to the other, so we declare $q_1 \sim q_2$ when we can cycle q_2 into q_1. The reader may check that this forms an equivalence relation.

8.2. Equivalence classes and partitions.

DEFINITION 8.3. Let A be a set and let $\sim$ be an equivalence relation on A. For $a \in A$, the *equivalence class* of a, written $[a]_\sim$ (or just $[a]$ if $\sim$ is clear from context) is the set

$$[a]_\sim := \{b \in A \mid a \sim b\}.$$

In the King Arthur problem, if $q = (1, 2, \ldots, 13)$, then $[q]_\sim$ is the set of permutations that can be rotated into q. For instance, $(2, 3, \ldots, 13, 1) \in [q]_\sim$.

More generally, think of the elements of a set as the residents of an apartment complex. Declare two elements equivalent if they live together. Then the equivalence classes are naturally in bijection with the apartments in the apartment building: we can think of an equivalence class as the set of people inhabiting a particular apartment.[3] The following theorem sharpens this analogy.

[3]This is true under mild hypotheses on the apartment building: every apartment has at least one resident, and no residents live in more than one apartment.

THEOREM 8.4. *If A is a set and $\sim$ is an equivalence relation on A, then for all $a, b \in A$*

(1) $a \in [a]$,
(2) if $a \sim b$, *then* $[a] = [b]$,
(3) if $a \not\sim b$, *then* $[a] \cap [b] = \varnothing$, *and*
(4) $\bigcup_{a \in A} [a] = A$.

Some comments on the notation are in order. First, $a \not\sim b$ simply means that (a, b) is not an element of $\sim$. Second, the indexed union $\bigcup_{a \in A} [a]$ may look intimidating, but it just means that we take the union of all the sets $[a]$ where a runs through A.

PROOF. (1) Since $\sim$ is reflexive, $a \sim a$ and thus $a \in [a]$.
(2) Suppose $a \sim b$ and $c \in [a]$. Then, by definition, $a \sim c$. Furthermore, symmetry tells us that $b \sim a$. Thus transitivity (applied to $b \sim a$, $a \sim c$) implies that $b \sim c$, *i.e.*, $c \in [b]$. This proves that $[a] \subseteq [b]$. The reader may now write down a nearly identical proof that $[b] \subseteq [a]$, whence $[a] = [b]$.
(3) Suppose $a \not\sim b$. We must show that if $c \in [a]$, then $c \notin [b]$. Suppose for contradiction that $c \in [a]$ and $c \in [b]$. Then $a \sim c$ and $b \sim c$. By symmetry and transitivity, we learn that $a \sim b$, a contradiction. We conclude that if $a \not\sim b$, then $[a] \cap [b] = \varnothing$.
(4) Since each $[a]$ is a subset of A, we know that $\bigcup_{a \in A} [a] \subseteq A$. The opposite inclusion follows from (1): if $b \in A$, then $b \in [b]$, and thus $b \in \bigcup_{a \in A} [a]$ because $[b]$ is one of the terms in the indexed union.

□

Properties (3) and (4) of equivalence classes in Theorem 8.4 tell us that equivalence classes form a "partition," a concept which deserves its own definition.

DEFINITION 8.5. A family of subsets $P_i \subseteq A$, where i ranges through an index set I, is a *partition* of A if $i \neq j \in I$ implies that $P_i \cap P_j = \varnothing$ and $\bigcup_{i \in I} P_i = A$.

Going back to our apartment complex analogy, we have a set of residents in the building A and then sets P_i of residents in apartment i for each $i \in I$, where I is the set of apartments.

We have seen that an equivalence relation on a set A produces a partition of A into equivalence classes. The converse is true as well: each partition produces an equivalence relation on A.

THEOREM 8.6. *Suppose $\mathcal{P} = \{P_i \subseteq A \mid i \in I\}$ is a partition of A. Define a relation $\sim$ on A where $a \sim b$ if and only if there exists $P_i \in \mathcal{P}$ such that both a and b belong to P_i. Then $\sim$ is an equivalence relation.*

PROOF. We first check that $\sim$ is reflexive. Given $a \in A$, we know that a is in some P_j, $j \in I$ because $\bigcup_{i \in I} P_i = A$. Thus $a \sim a$.

The definition of $\sim$ does not depend on the order of a and b, so $\sim$ is clearly symmetric: $a \sim b$ implies that $b \sim a$.

For transitivity, simply note that if both a and b are in P_i, and both b and c are in P_i, then a and c are in P_i. Thus $a \sim b$ and $b \sim c$ implies that $a \sim c$. □

The reader may check[4] that the constructions of this section give us a bijection between equivalence relations on A and partitions of A.

Since we are studying combinatorics in this class, it is only natural to ask how many partitions there are on A when $|A| < \infty$. This is a surprisingly subtle question, and we're not quite ready to develop the answer yet (but give it a try if you want to!).

[4]One of the most dangerous phrases in mathematical writing! You really should check when you see this, as it is too often a standin for "The author is too lazy to check."

8.3. Enumerating equivalence classes. Thinking about King Arthur's Round Table again, we see that we are trying to enumerate (count) the number of equivalence classes on Q, the set of queuings, with respect to the rotation equivalence relation $\sim$. The set of equivalence classes gets its own special notation: $Q/\sim$. We can reinterpret the argument from the introduction as saying that each equivalence class is of size 13. Thus the total number of equivalence classes is

$$|Q/\sim| = |Q|/13 = 13!/13 = 12!.$$

This is a general counting principle: If A is a set equipped with an equivalence relation $\sim$, and each of the $\sim$ equivalence classes has size m, then

$$|A/\sim| = |A|/m.$$

There is another way to count equivalence classes that we can again illustrate with the Round Table, namely, the method of choosing representatives. Suppose we have a way of picking exactly one representative from each equivalence class in $A/\sim$. Then the total number of such representatives will be equal to $|A/\sim|$. How can we do this for the Round Table problem? Well, since we can rotate the table, let's always put King Arthur at the top of it. Within each equivalence class of seatings, exactly one has Arthur at the top, so that will do the trick. Once we've put Arthur at the top, there are 12 ways to fill the seat to his left, then 11 ways to fill the left to the left of that one, *etc.*, revealing that there are

$$12 \cdot 11 \cdot 10 \cdots 1 = 12!$$

such representatives. We conclude that there are $12!$ seatings ($|Q/\sim|$) as well.

Let's do one more familiar example through the lens of equivalence relations. Consider the word $OUROBOROS$. (The ouroboros is an alchemical symbol for infinity in which a snake eats its own tail.) How many distinct strings can we make from the letters in $OUROBOROS$? We approach this by enumerating a larger set and then putting an equivalence relation on it so that the equivalence classes correspond to the distinct strings.

Let P be the set of permutations of the nine symbols $O_1, U, R_1, O_2, B, O_3, R_2, O_4, S$. We see that $|P| = 9!$. For $p, q \in P$, declare that $p \simeq q$ when p and q produce the same string after forgetting the subscripts. (For instance, $O_1O_2UO_3OR_1O_4R_2BS \simeq O_3O_4UO_2R_2O_1R_1BS$ because $OOUORORBS = OOUORORBS$.) If we can count $|P/\simeq|$, then we will have counted the number of distinct strings made from the letters in $OUROBOROS$. To this end, note that each equivalence class contains $4! \cdot 2! = 48$ permutations. (This is the number of ways to reorder the four O's and two R's.) Thus our first counting principle tells us there are $|P/\simeq| = 9!/48 = 7560$ strings.

9. Day 7

PROBLEM 9.1. For the following relations (with their standard meanings), determine what (if any) of the three properties of an equivalence relation they have: $\neq$, $>$, $\leq$.

PROBLEM 9.2. Consider the relation $\sim$ on $\mathbb{R}$ such that $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

PROBLEM 9.3. How many ways can we string n distinct beads on a necklace? We say that two lists of the n beads are equivalent if each bead is adjacent to the same two beads on each list. (The first and last beads on the list are considered adjacent.)

(a) Prove that the above relation on bead lists is an equivalence relation.
(b) How many lists are in an equivalence class?
(c) How many equivalence classes are there?

PROBLEM 9.4. Use an equivalence class count to interpret and answer the following question: n Americans and n Russians attend a meeting and sit around a round table. If Americans and Russians alternate seats, in how many ways may they be seated?

PROBLEM 9.5. We place two red and two black checkers on the corners of a square. Say that two configurations are equivalent if one can be rotated to the other. Check that this is an equivalence relation, and write down its equivalence classes. Can the number of equivalence classes be found by dividing 6 (the number of words in RRBB) by some natural number?

10. Day 8

Recall that for natural numbers n, k, the number

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!},$$

read "n choose k," is the number size k subsets of an n-element set. If $n \geq k$, this can also be written as $\frac{n!}{k!(n-k)!}$.

PROBLEM 10.1. Compute the sums

$$\begin{aligned}
&\binom{1}{0}\\
&\binom{2}{0} + \binom{2}{2}\\
&\binom{3}{0} + \binom{3}{2}\\
&\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\\
&\binom{5}{0} + \binom{5}{2} + \binom{5}{4}\\
&\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}\\
&\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6}
\end{aligned}$$

and develop a conjecture regarding the value of

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

where the sum's final term is $\binom{n}{n-1}$ or $\binom{n}{n}$ depending on whether n is odd or even, respectively. Give a combinatorial argument proving that your conjecture is true.

PROBLEM 10.2. Compute the sums

$$\begin{aligned}
&\binom{0}{0}^2\\
&\binom{1}{0}^2+\binom{1}{1}^2\\
&\binom{2}{0}^2+\binom{2}{1}^2+\binom{2}{2}^2\\
&\binom{3}{0}^2+\binom{3}{1}^2+\binom{3}{2}^2+\binom{3}{3}^2\\
&\binom{4}{0}^2+\binom{4}{1}^2+\binom{4}{2}^2+\binom{4}{3}^2+\binom{4}{4}^2\\
&\binom{5}{0}^2+\binom{5}{1}^2+\binom{5}{2}^2+\binom{5}{3}^2+\binom{5}{4}^2+\binom{5}{5}^2
\end{aligned}$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

Give a combinatorial argument proving that your conjecture is true.

11. Day 9

PROBLEM 11.1. How many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.) Develop a conjecture and prove it.

PROBLEM 11.2. Use algebra and the binomial theorem to prove that

$$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2.$$

PROBLEM 11.3. Use a combinatorial argument and an algebraic argument to produce two proofs of the identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}=\binom{n}{m}2^{n-m}.$$

[*Hint for the algebraic case*: First prove that $\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}$.]

12. Day 10

PROBLEM 12.1. The 0-th diagonal in Pascal's triangle is the constant sequence of 1's. The first diagonal is the sequence of positive integers $1,2,3,\ldots$. What is the second diagonal? The third? The n-th?

PROBLEM 12.2. You proved in your homework that $n^2=\binom{n}{2}+\binom{n+1}{2}$. Where do these terms appear in Pascal's triangle? Use your "second diagonal" interpretation from Problem 1 to produce a new proof of this identity.

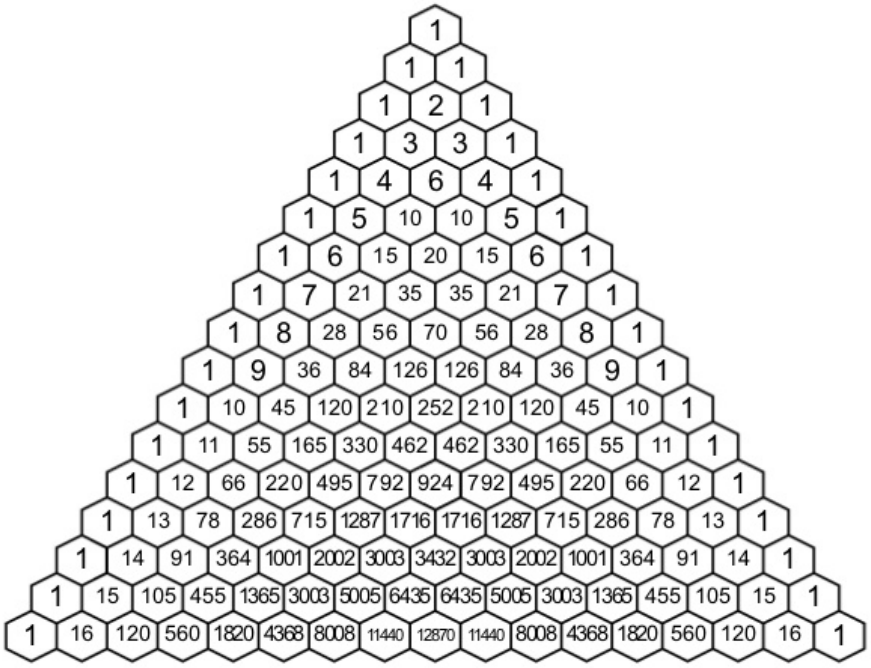


FIGURE 3. Pascal's triangle, 0-th through 16-th rows.

PROBLEM 12.3. How many odd numbers are there in the 2020-th row of Pascal's triangle? (To answer this, you may as well find a general formula for the number of odd numbers in the n-th row of Pascal's triangle. [*Hint*: How many odd numbers in the 2^k-th row?])

13. Day 11

PROBLEM 13.1. Use induction to show that

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$$

for $n \geq 1$.

PROBLEM 13.2. Use induction to prove that the number of permutations of $\underline{n} = \{1, 2, \ldots, n\}$ is $n!$.

PROBLEM 13.3. Use induction to prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for $n \geq 1$.

PROBLEM 13.4. Use induction to prove that a convex n-gon has $n(n-3)/2$ diagonals.

PROBLEM 13.5. Use induction to prove that

$$\binom{2n}{n} < 2^{2n-2}$$

for $n \geq 5$.

14. Day 12

The inclusion-exclusion principle tells us how to count the size of a union of sets. Its first two cases are

$$|A\cup B| = |A|+|B|-|A\cap B| \qquad \text{and} \qquad |A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|.$$

The general formula is messier, but is underpinned by the same idea of counting, removing duplicate count, adding back in things removed too many times, *etc*.

THEOREM 14.1 (Inclusion-Exclusion Principle). *Suppose $A_1, A_2, \ldots, A_n$ are finite sets. Then*

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots A_n| = &\sum_{1\le i_1\le n} |A_{i_1}| - \sum_{1\le i_1<i_2\le n} |A_{i_1}\cap A_{i_2}| + \cdots \\ &+ (-1)^{k-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} |A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}| + \cdots \\ &+ (-1)^{n-1}|A_1\cap A_2\cap\cdots\cap A_n|.\end{aligned}$$

This can be equivalently phrased as

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\varnothing\ne J\subseteq \underline{n}} (-1)^{|J|-1} \left|\bigcap_{i\in j} A_i\right|.$$

PROBLEM 14.2. At a large university, 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken a course in both Spanish and French, 23 have taken a course in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

PROBLEM 14.3. How many poker hands (5 cards) from a regular deck (52 cards) have at least one card from each of the four standard suits? *Hint*: Let $N_\spadesuit$ be the collection of hands containing no spades, and similarly define $N_\clubsuit$, $N_\heartsuit$, and $N_\diamondsuit$. What is the relationship between the answer to this question and $|N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|$?

15. Day 13

The *pigeonhole principle* tells us that if we have n pigeonholes and $k > n$ pigeons, then if we put all the pigeons in pigeonholes, one of the pigeonholes must contain at least two pigeons. In the language of functions, this says that if $f : A \to B$ is a function with $|A| > |B|$, then f is *not* injective. (Careful! It does not say that f is surjective — make sure you appreciate the difference.)

The *generalized pigeonhole principle* says that if there are n pigeonholes and $k > rn$ pigeons where r is a positive integer, then if we put all the pigeons in pigeonholes, one of the pigeonholes must contain at least $r + 1$ pigeons. This is equivalent to the statement that if N objects are put in b boxes, then some box contains at least $\lceil N/b\rceil$ objects.

PROBLEM 15.1. In a round robin chess tournament with n participants, every player plays every other player exactly once. Prove that at any given time during the tournament, two players have finished the same number of games.

PROBLEM 15.2. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be given distinct 10-digit telephone numbers of the form NXX-NXX-$XXXX$ where each X is any digit from 0 to 9 and each N represents a digit from 2 to 9? (The area code is the first three digits.)

PROBLEM 15.3. Show that in the sequence 7, 77, 777, 7777, ... there is an integer divisible by 2003. (*Hint*: First use "obvious" facts about integer divisibility to prove that if there are terms in the sequence $a_i > a_j$ such that $a_i - a_j$ is divisible by 2003, then there is a term of the sequence divisible by 2003. In order to show that such a_i, a_j exist, note that $a_i - a_j$ is divisible by 2003 if and only if a_i and a_j have the same remainder upon division by 2003; then use the pigeonhole principle.)

16. Derangements

Imagine n suitors all trying to woo each other. They each purchase a bouquet of flowers, and proceed to give the bouquet to their beloved. Assume further that the suitors are in the fortunate situation that no two suitors have the same beloved, and also that no suitor is so narcissistic as to have themself as beloved. In how many ways might the suitors distribute their bouquets?

We rephrase this problem mathematically as follows: an assignment of bouquets is a function $f : \underline{n} \to \underline{n}$ where $\underline{n} = \{1, 2, \ldots, n\}$. Since no two suitors have the same beloved, the function is injective, and thus surjective as the domain and codomain have the same cardinality. Thus f is a permutation. Finally, the non-narcissism clause guarantees that $f(i) \neq i$ for all $i \in \underline{n}$. When $f(i) = i$, we call i a *fixed point* of f, so we are looking for permutations of $\underline{n}$ with no fixed points. Such permutations are called *derangements*. The problem of enumerating derangements was first posed by Pierre de Montmort in 1708, and subsequently resolved independently by de Montmort and Nicholas Bernoulli in 1713.

The number of derangements of $\underline{n}$ is called the *subfactorial* of n and is denoted $n¡$. (Other notations include $!n$, $D(n)$, and D_n, but we will use the inverted exclamation point. While typically used at the start of exclamatory Spanish-language sentences, in 1668, John Wilkins proposed the punctuation ¡ at the end of a sentence to denote irony.)

In order to count $n¡$, we will count the "bad" permutations of $\underline{n}$ with at least one fixed point. For $i \in \underline{n}$, let A_i denote the set of permutations of $\underline{n}$ with i as a fixed point. Then $n¡ = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$. We aim to count $|A_1 \cup \cdots \cup A_n|$ via the inclusion-exclusion principle.

Note that $|A_i| = (n-1)!$. Indeed, for $f \in A_i$, $f(i) = i$ and f is free to permute the other $n-1$ elements of $\underline{n}$. What about $|A_i \cap A_j|$, $i \neq j \in \underline{n}$? If $f \in A_i \cap A_j$, then $f(i) = i$ and $f(j) = j$, but f is free to permute the other $n-2$ elements of $\underline{n}$, so $|A_i \cap A_j| = (n-2)!$. Similarly, if $i_1 < i_2 < \cdots < i_k$, then $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$. Since there are $\binom{n}{k}$ k-fold intersections, and each has the same cardinality $(n-k)!$, inclusion-exclusion implies that

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}.$$

Thus

$$n¡ = n! - |A_1 \cup \cdots \cup A_n| = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}.$$

By factoring out $n!$ (and replacing 1 with $\frac{1}{0!}$), we can further rewrite this as

$$n¡ = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

This gives us a formula for the number of derangements of $\underline{n}$, and also a count for our initial problem regarding distribution of bouquets.

If you have taken calculus, you may recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus when $n \to \infty$,

$$\frac{n¡}{n!} \longrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \approx 0.36788.$$

The convergence of this series is quite rapid, and it is actually the case that $n¡$ is the integer nearest $n!/e$ for all $n > 0$.

This is quite remarkable! About 0.36788 of permutations of $\underline{n}$ are derangements, independent of n. For reference, here is a table listing n, $n¡$, and the approximate value of $n!/e$.

n	$n¡$	$n!/e$
1	0	0.36788
2	1	0.73576
3	2	2.20723
4	9	8.82911
5	44	44.1455
6	265	264.873
7	1854	1854.11
8	14833	14832.9

17. Day 14

Recall that a *derangement* is a fixed point-free permutation (meaning $\pi(i) \neq i$ for all i) and that the number of derangements of an n-element set is

$$n¡ = n!(1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n/n!).$$

PROBLEM 17.1. How many derangements π of $\underline{n}$ have $\pi(1) = 2$ and $\pi(2) = 1$? Fix some k, $2 \leq k \leq n$; how many derangements π of $\underline{n}$ have $\pi(1) = k$ and $\pi(k) = 1$?

PROBLEM 17.2. How many derangements π of $\underline{n}$ have $\pi(1) = 2$ and $\pi(2) \neq 1$? Fix some k, $2 \leq k \leq n$; how many derangements π of $\underline{n}$ have $\pi(1) = k$ and $\pi(k) \neq 1$?

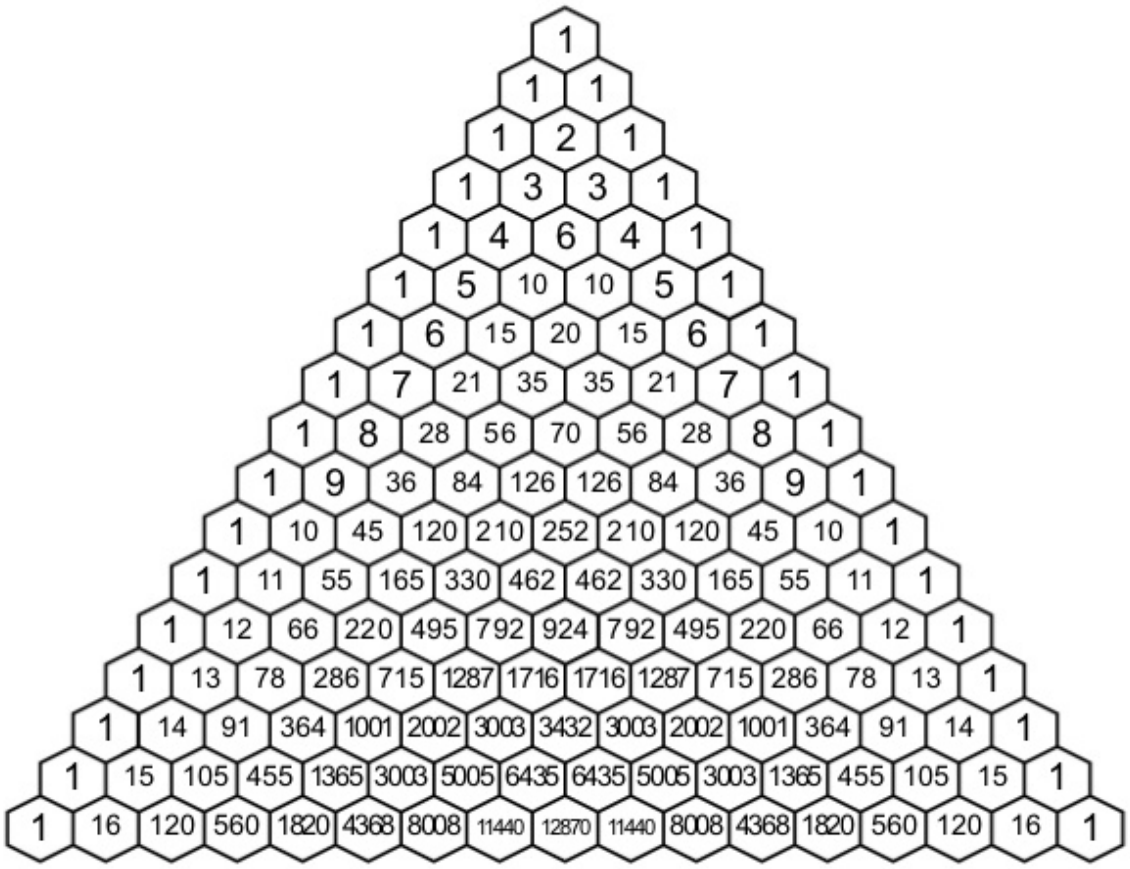
PROBLEM 17.3. Let $n¡$ be the number of derangements of $\underline{n}$. Use your answers to Problems 1 and 2 to find a formula for $n¡$ in terms of $(n-2)¡$ and $(n-1)¡$. Determine $1¡$ and $2¡$ by hand and then use your formula to determine $n¡$ for $n = 3$, 4, 5, and 6; check that your answers match with the closed formula given by the inclusion-exclusion principle.

18. Day 15

PROBLEM 18.1. In how many ways can you fill a $2 \times n$ chessboard with 2×1 dominoes? (Each domino must cover exactly two squares, but may be placed horizontally or vertically.) Work out the answer directly for several small values of n, make a conjecture about the overall pattern, then prove your conjecture.

PROBLEM 18.2. Mark the first entry in some row of Pascal's triangle (this is a 1). Move one step east and one step northeast, and mark the entry there. Repeat this until you exit the triangle. Compute the sum of the entries you marked.

(a) Repeat this process for several other rows of Pascal's triangle. Guess what pattern is emerging.
(b) Express your guess in terms of a sum of binomial coefficients and prove that it is true.



PROBLEM 18.3. Extend the Fibonacci sequence backwards (with negative indices) via the relation $F_n = F_{n+2} - F_{n+1}$. Write out the terms $F_{-5}, F_{-4}, F_{-3}, \ldots, F_3, F_4, F_5$ (and maybe a few more in either direction). Come up with a conjecture about the relation between Fibonacci numbers with negative indices and positive indices. Prove your conjecture.

19. Day 16

PROBLEM 19.1. Compute the following sums:

$$F_1$$
$$F_1 + F_3$$
$$F_1 + F_3 + F_5$$
$$F_1 + F_3 + F_5 + F_7$$
$$F_1 + F_3 + F_5 + F_7 + F_9$$

Develop and prove a conjecture about the value of $G_n = \sum_{k=1}^{n} F_{2k-1}$.

PROBLEM 19.2. Develop and prove a conjecture about the value of $F_{n-1}F_{n+1} - F_n^2$.

20. Day 17

PROBLEM 20.1. In this problem we will determine the number of regions in the plane created by a system of n mutually overlapping circles in general position. By *mutually overlapping*, we

mean that each pair of circles intersects in two distinct points. By *general position*, we mean that there are no three circles through a common point. Let a_n be the number of regions created by such a system.

(a) Draw some pictures to determine a_0, a_1, a_2, and a_3.
(b) Do you have a conjecture regarding the value of a_n? Check it by drawing a picture to determine a_4.
(c) Take a system of $n-1$ circles (creating a_{n-1} regions) then add an n-th circle which is mutually overlapping and in general position. How many times does this circle intersect circles in the system of $n-1$ circles? How many arcs on the new circle are created by these intersections?
(d) Use your above analysis to determine a recurrence relation which a_n satisfies. (For which n does the recurrence relation hold?)
(e) Use your recurrence relation to find a closed formula (only in terms of n) for a_n (at least for n sufficiently large).
(bonus) Can you find a direct (as opposed to recurrence-based) argument for your formula in (e)?

PROBLEM 20.2. An anxious ant wanders through a 3×3 grid of the form

1	2	3
4	5	6
7	8	9

and only passes between cells via edges (as opposed to corners). We would like to count the number p_n of length n paths the ant can take where there is no constraint on where the ant starts or ends the path. (A "step" in the path is when the ant changes cells, despite the fact that this takes the ant many many steps. We do not permit the "stay put" step.) A direct recurrence relation on p_n is difficult to come by. (If the $(n-1)$-th step is to cell 1, then the ant can only travel to 2 or 4, but if the $(n-1)$-th step is to cell 5, the ant can travel to 2, 4, 6, or 8.) Instead, we seek multiple recurrence relations (and some good luck).

(a) Let a_n denote the number of length n paths ending in 1, let b_n denote the number of length n paths ending in 2, and let c_n denote the number of length n paths ending in 5. What is the relationship between p_n and these three sequences. (Use symmetry!)
(b) Determine a *system of recurrence relations* for the sequences a_n, b_n, c_n. (This is like a recurrence relation, but each sequence may depend on previous terms of the other sequences.)
(c) Use algebra to find a recurrence relation for b_n (only in terms of previous terms from the same sequence).
(d) Put everything together to get a recurrence relation for p_n.
(e) Compute p_0, p_1, p_2, p_3, p_4, and p_5. Why is the ant anxious?

21. Day 18

PROBLEM 21.1. A *complete graph* on n vertices, denoted K_n, has every possible edge. Draw pictures of K_3, K_4, and K_5. How many edges are there in a complete graph on n vertices? For a general graph $G = (V, E)$, make an inequality relating $|V|$ and $|E|$.

PROBLEM 21.2. A graph $G = (V, E)$ is called *bipartite* if $V = A \cup B$ with $A \cap B = \varnothing$ and there are no edges between vertices in A and similarly for B (so only edges between a vertex in A and a vertex in B are allowed). The *complete bipartite graph on $p+q$ vertices*, denoted $K_{p,q}$, has $|A| = p$, $|B| = q$, and all possible edges between A and B.

(a) Draw pictures of $K_{2,3}$ and $K_{3,5}$.
(b) How many edges are in $K_{p,q}$?

(c) If $|A| = p$ and $|B| = q$ with $A \cap B = \varnothing$, how many (not necessarily complete) bipartite graphs have vertex set $A \cup B$?

PROBLEM 21.3. Suppose $G = (V, E)$ and $G' = (V', E')$ are graphs.

(a) When should a function $f : V \to V'$ be considered a "map" $G \to G'$?
(b) When should we consider G and G' to be "the same" graph?

22. Day 19

PROBLEM 22.1. Let $G = (V, E)$ be a graph with connected subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ such that $V_1 \cap V_2 \neq \varnothing$. Prove that G is connected.

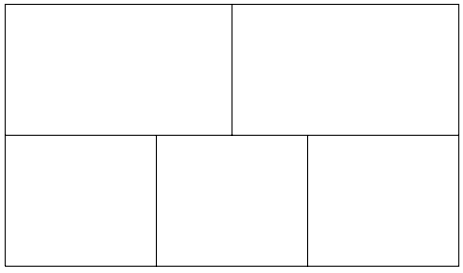
Call a graph *acyclic* if it does not contain any subgraphs which are cycles. A *tree* is a connected acyclic graph. A disconnected acyclic graph is called a *forest*.

PROBLEM 22.2. How many edges are there in a tree with n vertices? Prove your assertion (by induction?).

PROBLEM 22.3. Prove that a graph G is a tree if and only if there is a *unique* path between any two vertices of G.

23. Day 20

Consider the following floor plan for a building:



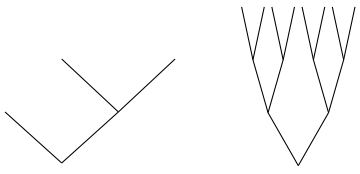
We would like to know if it is possible to cross each interior wall in the building exactly once (without teleporting).

PROBLEM 23.1. (a) Turn this into a graph theory problem about a particular kind of walk.
(b) Either find such a walk, or prove that no such walk exists.
(c) What if we want to pass through the exterior walls as well?

24. Day 21

A *full binary tree* is a rooted tree in which each vertex has either two children or no children; furthermore, when there are two children, one is designated *left* and the other *right*. Vertices with no children are called *leaves*.

Here are some examples:

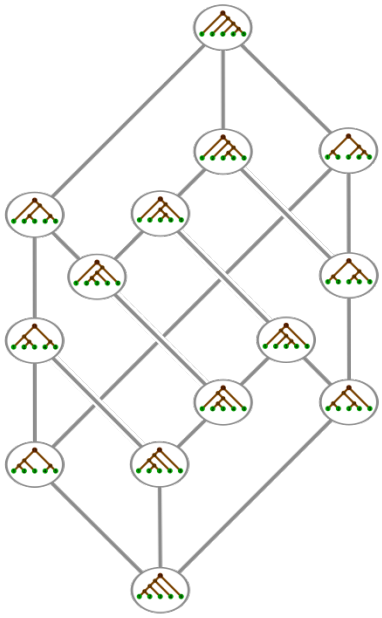


PROBLEM 24.1. Let C_n denote the number of unlabelled full binary trees with $n+1$ leaves. Prove that $C_0 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$. Compute the first several values of C_n and draw the corresponding full binary trees.

Here is an alluring picture of the 14 full binary trees with 5 leaves. Do you see what the edges represent?



The numbers C_n are called the *Catalan numbers* and can be expressed concisely as $C_n = \frac{1}{n+1}\binom{2n}{n}$. The standard proof of this fact uses *generating functions* and will not be presented here. A bijective proof for this formula appears after we establish that some additional combinatorial structures counted by Catalan numbers.

PROBLEM 24.2. Find an explicit bijection between full binary trees with $n+1$ leaves and full parenthesizations of $n+1$ factors. (For instance, the full parenthesizations of abc are $(ab)c$ and $a(bc)$, while the full parenthesizations of $abcd$ are $((ab)c)d$, $(a(bc))d$, $(ab)(cd)$, $a((bc)d)$, and $a(b(cd))$.) This proves that C_n counts the number of full parenthesizations of $n+1$ factors.

It is also the case that C_n is the number of ways of arranging n pairs of correctly matched parentheses. (Can you prove it?) This perspective is very important in computer science, where trees are frequently stored via bracketing schemes.

25. Day 22

PROBLEM 25.1. A *Dyck path* of length $2n$ is a monotonic lattice path in $[0,n]^2$ starting from $(0,0)$ and ending at (n,n) which never goes above the diagonal. Prove that there are C_n Dyck paths of length $2n$.

Dyck paths also give a proof of the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

PROOF. Recall that there are $\binom{2n}{n}$ monotonic lattice paths from $(0,0)$ to (n,n). We aim to partition the monotonic paths into $n+1$ subsets of equal size, where precisely one of the subsets is the collection of Dyck paths. This will prove that $C_n = \binom{2n}{n}/(n+1)$, as desired.

We define the *exceedance* of a monotonic lattice path to be its number of vertical steps above the diagonal. The exceedance of a monotonic lattice path from $(0,0)$ to (n,n) is between 0 and n (inclusive), and the Dyck paths are precisely those monotonic lattice paths with exceedance 0. Let P be the set of monotonic lattice paths from $(0,0)$ to (n,n) and let E_i be the set of paths with exceednace i; then $P = E_0 \cup E_1 \cup \cdots \cup E_n$ is clearly a partition of P. If we can show that $|E_0| = |E_1| = |E_2| = \cdots = |E_n|$, then we will be done.

Given a path $p \in E_i$, write $p = BrAuC$ where r is the first right step below the diagonal and u is the first up step touching the diagonal after r. Then B is a path above the diagonal with exceedance $j \leq i$, A is a path below the diagonal, and C is the reamining path with exceedance $i-j$. Switch Br and Au to produce $f(p) = AuBrC$. The exceedances of A, uBr, and C are 0, $j+1$, and $i-j$, respectively. (Draw some pictures and check this!) Thus $f(p) \in E_{i+1}$.

Given a path $q \in E_{i+1}$, write $q = AuBrC$ where u is the first up step above the diagonal and r is the first right step touching the diagonal after u. Define $g(q) = BuAdC$ and check that $g(q) \in E_i$. Finally, check that $f : E_i \to E_{i+1}$ and $g : E_{i+1} \to E_i$ are inverse to each other. □

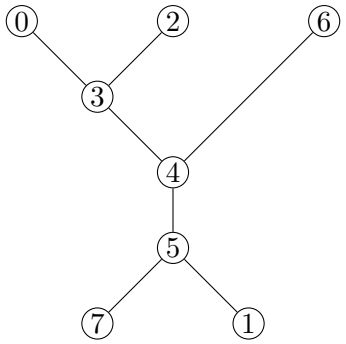
PROBLEM 25.2. Prove that we can also express C_n as

$$C_n = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n+1}$$

26. Day 23

A *leaf* of a tree is a vertex of degree 1. Suppose T is a tree with vertex set $\{0, 1, 2, \ldots, n-1\}$. The *Prüfer code* of T is the sequence of length $n-2$ with entries in $\{0, 1, \ldots, n-1\}$ generated by the following algorithm: At step i, remove the leaf with the smallest label not equal to 0 and set the i-th entry of the Prüfer code equal to the label of the leaf's neighbor. After step $n-2$, the end of the algorithm, one is left with a single edge joining some node to 0.

For instance, the Prüfer code of the following graph is 534543.



In your reading, you learned how to turn a Prüfer code into a tree by writing down its extended Prüfer code, a $2 \times n$ array with entries in $\{0, 1, \ldots, n-1\}$ with columns corresponding to edges. To quote,

> Each entry in the first row of the extended Prüfer code is the smallest integer that does not occur in the first row before it, nor in the second row below or after it.

One applies this procedure with initial data the second row consisting of the Prüfer code with a 0 tacked on the end.

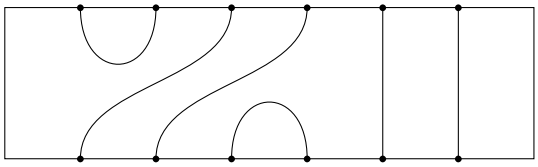
PROBLEM 26.1. Draw a tree on vertex set $\{0, 1, \ldots, n-1\}$ with $n = 6$, 7, 8, or 9. Determine its Prüfer code and write the Prüfer code on the whiteboard. Then trade Prüfer codes with another group and decode into a tree. Draw the tree next to its Prüfer code and check your work with the group that made the Prüfer code.

PROBLEM 26.2. Which trees have Prüfer codes that contain only one value?

PROBLEM 26.3. Which trees have Prüfer codes with distinct values in all positions?

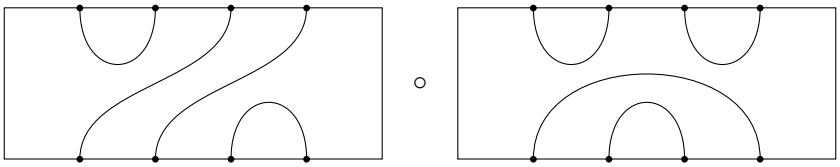
27. Day 24

Place n dots along the top of a rectangle, and place n dots along its bottom. Now draw n non-crossing strings in the box which connect distinct points. Such a configuration is called a *Temperley-Lieb diagram* on $2n$ nodes. Here is a Temperley–Lieb diagram on 12 nodes:

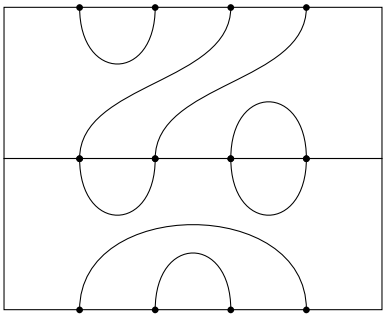


PROBLEM 27.1. Show that there are C_n Temperley–Lieb diagrams on $2n$ nodes.

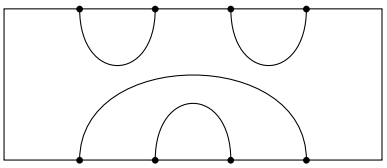
We can *compose* two Temperley–Lieb diagrams (on the same number of nodes) by placing one on top of the other and gluing the strings together. This results in a new Temperley–Lieb diagram, but possibly with some loops floating around in it. If there are k loops, we make the rule of deleting all loops and placing a formal monomial q^k next to the diagram. For instance, the composite



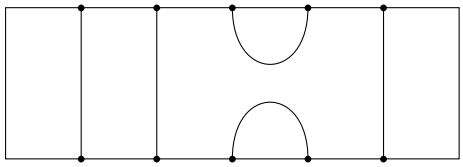
is computed as



which is then reinterpreted as


$\cdot q^1$.

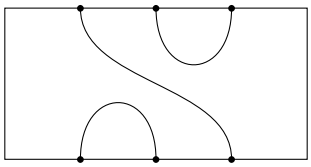
PROBLEM 27.2. Let TL_n denote the set of Temperley–Lieb diagrams on $2n$ nodes. For $1 \leq i \leq n-1$, let U_i be the Temperley–Lieb diagram with all vertical strings except for a cup and a cap joining the i-th and $(i+1)$-th nodes on the top and bottom. For instance, here is U_3 in TL_5:


.

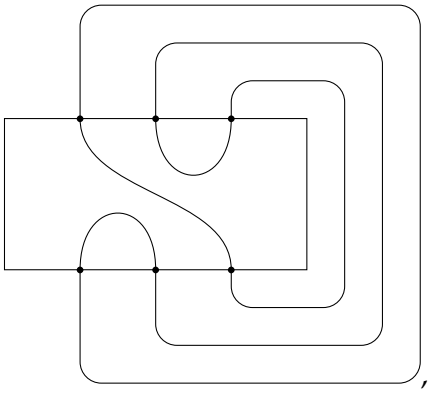
Let 1 denote the diagram with all vertical strings.

(a) Observe that 1 is a 2-sided identity for composition of Temperley–Lieb diagrams.
(b) Draw pictures to show that the U_i satisfy the following relations:
   - » $U_i^2 = U_i q$ for all $1 \leq i \leq n-1$,
   - » $U_i U_{i+1} U_i = U_i$ for all $1 \leq i \leq n-2$,
   - » $U_i U_{i-1} U_i = U_i$ for all $2 \leq i \leq n-1$,
   - » $U_i U_j = U_j U_i$ for all $1 \leq i, j \leq n-1$ such that $|i-j| \neq 1$.
(c) Show that every Temperley–Lieb diagram can be written as a composition of $1, U_1, U_2, \ldots, U_{n-1}$.

The *Markov trace* is an operation on Temperley–Lieb diagrams which connects each dot on the top row to the corresponding dot on the botoom row using auxiliary loops (on the outside of the rectangle) and then records the number of loops, k, as q^k. For instance, the trace of



in TL_3 is computed by forming the extended diagram


,

counting that only one loop was formed, and concluding that the trace is q.

PROBLEM 27.3. (a) Determine the trace of $1 \in TL_n$.

(b) Determine the trace of $U_i \in TL_n$, $1 \leq i \leq n-1$.

(c) Fix k between 1 and n, inclusive. Let $C_{n,k}$ denote the number of Temperley–Lieb diagrams in TL_n with trace q^k. By Problem 1, $\sum_{k=1}^{n} C_{n,k} = C_n$. Find recurrent and closed formulæ for $C_{n,k}$.

CHAPTER 2

Probability

1. Probability spaces

DEFINITION 1.1. We think of a **sample space** S as the set of all possible outcomes of an 'experiment' or observation. An **outcome** is an element of the sample space.

EXAMPLE 1.2. If we are rolling a 6-sided die, $S = \{1, 2, 3, 4, 5, 6\}$. If we are flipping a coin two times, $S = \{HH, HT, TH, TT\}$. If we are playing Minesweeper, $S = \{Die, LiveDie, LiveLiveDie, LiveLiveLiveD$

DEFINITION 1.3. An **event** E is a subset of the sample space, thought of as a collection of outcomes.

EXAMPLE 1.4. When we are rolling a 6-sided die, if E is rolling an even number, then $E = \{2, 4, 6\}$. If $H = \{4, 5, 6\}$, then one way to describe H is 'rolling higher than 3.'

There may be more than one way to describe the same event, and the same description might correspond to different events if the sample space is different.

Since events are sets, we can do the usual things to them.

DEFINITION 1.5. The **union** of two events A, B is the event $A \cup B$, which can be described as 'A or B.' The **intersection** of A, B is $A \cap B$, 'A and B.' The **complement** of A is A^c, 'not A,' or 'A doesn't happen.'

DEFINITION 1.6. $\emptyset$ is called the **null event** (it never happens) and S is the **certain event** (it always happens).

DEFINITION 1.7. Two events A, B are called **mutually exclusive** if $A \cap B = \emptyset$.

DEFINITION 1.8. Given a sample space S, a **probability distribution** is a map

$$P : \{events\} \longrightarrow [0, 1]$$

such that

i) $P(S) = 1$, $P(\emptyset) = 0$
ii) If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

We will usually call $P(E)$ the probability of E.

DEFINITION 1.9. A sample space along with a probability distribution is called a **probability space**. If every outcome is equally likely, it is called a **uniform probability space**. In a uniform probability space where $|S| < \infty$, $P(E) = |E|/|S|$.

Some properties of probability distributions follow directly from set theory!

PROPOSITION 1.10. i) If $A \subseteq B$, $P(A) \leq P(B)$.
ii) $P(A) = 1 - P(A^c)$
iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
iv) $P(A \cup B) + P(A^c \cap B^c) = 1$
v) $P(A \cap B) + P(A^c \cup B^c) = 1$

Here is how we might prove i):

PROOF. Suppose $A \subseteq B$. Then $A \cap B \backslash A = \emptyset$. So $P(B) = P(A) + P(B \backslash A)$. But $P(B \backslash A) \geq 0$, so $P(B) \geq P(A)$. □

EXAMPLE 1.11. Suppose we have a standard deck of 52 cards, with 13 cards of each suit: hearts ♡ and diamonds ♢ (both red), and clubs ♣ and spades ♠ (both black). Suppose we have shuffled the deck so that the cards are in random order, and we pick two cards off the top. What is the probability that the first two cards are both red?

Let's call R the event that the first two cards are red. The order of the cards is random, so any pair of cards is equally likely. Therefore $P(R) = |R|/|S|$. Here are two different ways to solve this problem; there are doubtless many more.

There are 52 possible first cards, and then 51 possible second cards, so the total number of outcomes is $52 \cdot 51$. There are 26 red cards, so there are $26 \cdot 25$ outcomes in R and $P(R) = \frac{26 \cdot 25}{52 \cdot 51} = \frac{25}{102}$.

Alternatively, there are $\binom{52}{2}$ ways to pick two distinct cards out of the deck. There are $\binom{26}{2}$ ways to pick red cards, so $P(R) = \frac{\binom{52}{2}}{\binom{26}{2}} = \frac{\frac{26!}{2!24!}}{\frac{52!}{2!50!}} = \frac{26 \cdot 25}{52 \cdot 51} = \frac{25}{102}$.

Just as in combinatorics, if you want to check your work, count it in two different ways and see if you get the same answer.

2. Day 25

Let S be our sample space (really any set) and let $\mathscr{E} = 2^S$ denote the corresponding collection of events (just the set of subsets of S). Recall that a *probability distribution* on S is a function

$$P : \mathscr{E} \to [0, 1]$$

such that (1) $P(S) = 1$, (2) $P(\varnothing) = 0$, and (3) if A, $B \in \mathscr{E}$ are mutually exclusive events (so $A \cap B = \varnothing$), then $P(A \cup B) = P(A) + P(B)$. If S is a finite set, then we can define the *uniform probability distribution* on S to be the function taking $A \subseteq S$ to $|A|/|S|$.

PROBLEM 2.1. A lottery has participants choose 5 distinct numbers from the set $\{1, 2, \ldots, 36\}$. On a prescribed date, the lottery announces a collection of 5 winning numbers. Complete the following prompts in order to determine why the lottery does not offer a prize for having selected only 1 winning number.

(a) What sample space is pertinent in this question? Describe it both as a collection of certain types of objects, and in a more mathematical fashion.
(b) Is it reasonable to put the uniform probability distribution on this sample space? (Assume that the lottery is fair.)
(c) Let B denote the event of choosing a ticket with no winning numbers. What $P(B)$?
(d) Let A denote the event of choosing a ticket with at least one winning number. What is $A \cap B$? $A \cup B$?
(e) Use the axioms for a probability distribution and your answer to (c) to determine $P(A)$.
(f) [Follow up question] Might it be reasonable to offer prizes for anyone with 2 or more winning numbers?

PROBLEM 2.2. What is the probability that in a random ordering of a standard deck of cards, the ace of spades precedes the king of hearts?

(a) Rephrase this as a question about permutations of $\underline{52}$. What is the sample space under consideration? the event?

(b) Prove that the probability of this event (under the uniform distribution) is $1/2$ by producing a bijection between the event and its complement. (Why does that solve things?)

PROBLEM 2.3. Your partner invites you to play a game: they write ten distinct real numbers on ten blank cards. The cards are shuffled randomly and placed face down on the table. You start at the top of the deck and start revealing cards. At any point you may choose to stop turning over cards and select the most recently revealed card. You win if your selection is the largest of all ten numbers (both those previously revealed and those still unrevealed). Devise a strategy which guarantees you will win this game at least 25% of the time.

3. Independence

Let's start with an example.

EXAMPLE 3.1. Alisha and Bachir each sit in a row of 7 chairs, choosing their seats at random. What is the probability that they don't sit next to each other?

There are $7 \cdot 6$ ways to sit. We could count all the different ways to sit so that there is at least one seat in between them. If A is in the first or last spot, B has 5 choices for where to sit. Otherwise B has only 4 choices, since A plus one seat on each side takes away 3 out of the 7 spots. Therefore there are $2 \cdot 5 + 5 \cdot 4 = 30$ different ways for the pair to sit not next to each other, and the probability of them not sitting next to each other is $\frac{30}{7 \cdot 6} = \frac{5}{7}$.

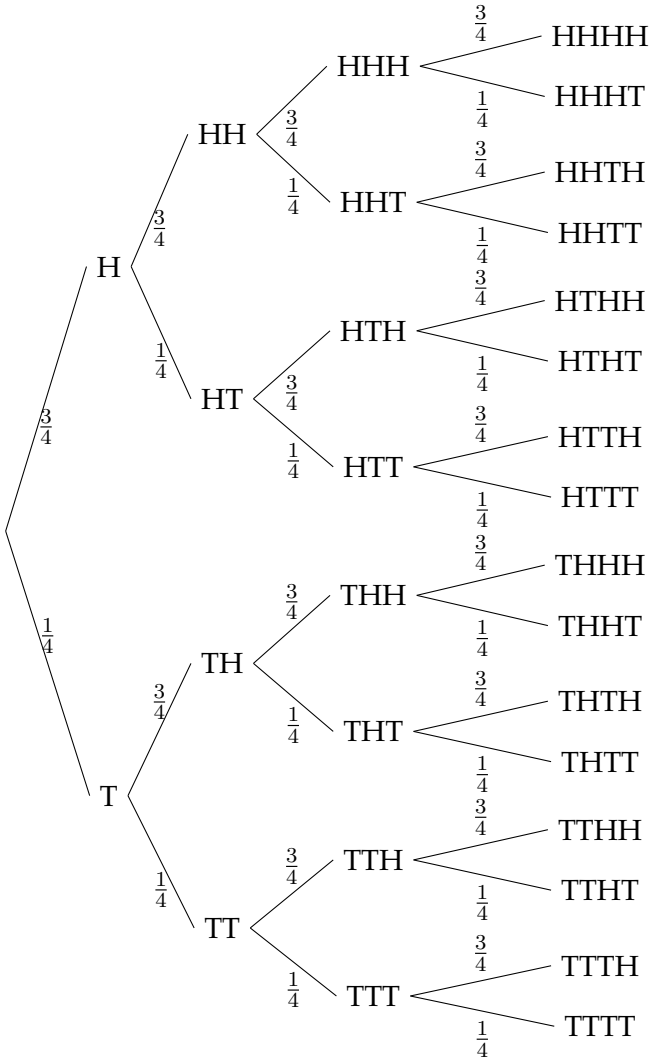
Alternatively, it's perhaps easier to count the different ways for them to sit together and then take the complement. In this case, there are 6 ways we can choose a spot for the pair and 2 ways they can sit in that spot (AB or BA) so the probability we want is $1 - \frac{6 \cdot 2}{7 \cdot 6} = 1 - \frac{2}{7} = \frac{5}{7}$.

Keep in mind that it's sometimes easier to count a complement. This can be a good way to check your answer.

DEFINITION 3.2. If $P(A \cap B) = P(A) \cdot P(B)$, then we call A and B **independent relative to** P. This is different than the colloquial meaning of independent. Unless independence is explicitly given in the problem, you have to prove it. Be suspicious of your intuition, because it is often wrong!

EXAMPLE 3.3. Suppose we have an unfair coin, so the probability of flipping heads is always 0.75. What is the probability of getting 4 heads in a row? 4 tails in a row? exactly 2 heads out of 4 flips?

Notice this is NOT a uniform probability space. However, each flip has the same probability of being heads as the flip before it. Effectively, the problem as stated is asserting that flipping heads on the first, second, third, or fourth flip are all independent of each other.
We can model this with what I call a probability tree. This is just a visual organizer, not a mathematical object. It's not the only way to solve this, but I like it! Each level in the tree will be an independent event, with branches labelled with probability. To calculate, find the right leaves, multiply the probabilities that go down to those leaves, and add them all up.



$$P(HHHH) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \left(\frac{3}{4}\right)^4 = \frac{81}{256}$$

$$P(TTTT) = \left(\frac{1}{4}\right)^4 = \frac{1}{256}$$

$$P(HHTT) + P(HTHT) + P(HTTH) + P(THHT) + P(THTH) + P(TTHH) = \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^2 \cdot 6$$
$$= \frac{54}{256} = \frac{27}{128}$$

Notice that we could also compute the last probability by $\binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2$.

EXAMPLE 3.4. Suppose that we draw a number from the set $\{1, 2, \ldots, 49\}$ at random. Let F be 'picking a number divisible by 5' and let E be 'picking an even number.' Are these events independent?

We can construct a uniform probability space to solve this, where $F = \{5, 10, \ldots, 45\}$ and $E = \{2, 4, \ldots, 48\}$. Then $|S| = 49$, $|F| = \lfloor \frac{49}{5} \rfloor = 9$, $|E| = \lfloor \frac{49}{2} \rfloor = 24$, and $|F \cap E| = \lfloor \frac{49}{10} \rfloor = 4$, so

$P(F) = \frac{9}{49}$, $P(E) = \frac{24}{49}$, $P(F) \cdot P(E) = \frac{9 \cdot 24}{49^2}$ and $P(F \cap E) = \frac{4}{49}$. But $\frac{4}{49} \neq \frac{9 \cdot 24}{49^2} = \frac{216}{2401}$, so these events are NOT independent.

This is how you prove that things are independent!

4. Day 26

PROBLEM 4.1. Show that if A and B are independent, then so are their complements A^c and B^c.

PROBLEM 4.2. We flip a fair coin n times. Let A be the event that the first coin flip was heads. Let B be the event that the number of heads was even. Let C be the event that the number of heads was more than the number of tails. Which pairs of these three events are independent?

PROBLEM 4.3. There are n players in a Go tournament. In this problem we will use probability theory to show that for certain n it is possible for every collection of 3 players there exists another player who has beaten them all.

(a) Suppose that the outcome of each game is random. (Perhaps the players are lazy and flip a coin to decide the winner.) Fix a 3-subset $\{x, y, z\}$ of players and some player w not in $\{x, y, z\}$. What is the probability that w wins against x, y, and z? What is the probability that w loses against at least one of x, y, z?
(b) Suppose we have another player w' different from w, x, y, and z. Are the results of w''s matches against x, y, z independent of the results of w's matches?
(c) How many players can appear in the role of w? What is the probability that each of them loses against at least one of x, y, z?
(d) Use your answer to (c) and the fact that there are $\binom{n}{3}$ 3-subsets of $\underline{n}$ to produce an upper bound on the probability that for at least one 3-subset $\{x, y, z\}$, no player beats x, y, and z simultaneously.
(e) What does it mean if your upper bound from (d) is less than 1? Use a computer to determine if there are n for which this happens.

5. Conditional probability

Recall that events $A, B \subseteq S$ are *independent* when $P(A)P(B) = P(A \cap B)$. What's happening when events are *not* independent?

DEFINITION 5.1. Let $A, B \subseteq S$ be events and assume $P(B) > 0$. Define $P(A|B) := P(A \cap B)/P(B)$. Then $P(A|B)$ is called a *conditional probability* and read "the probability of A given B."

Note that A and B with $P(B) > 0$ are independent if and only if $P(A|B) = P(A)$. Since $P(A|B)$ is the probability that A happens given that B happens, we see that A and B are independent when the occurrence of B does not make the occurrence of A any more or less likely.

EXAMPLE 5.2. We toss a fair coin four times. We don't see the results, but someone who does truthfully tells us that at least two of the tosses were heads. What is the probability that all four tosses were heads?

To answer this question, we must find $P(A|B)$ where A is the event "all four tosses are heads" and B is the event "at least two tosses are heads." Note that $A \cap B = A$, so $P(A|B) = P(A)/P(B)$. Of course, $P(A) = (1/2)^4 = 1/16$. Meanwhile, B is the disjoint union of the events "exactly two heads," "exactly three heads," and A. Thus

$$P(B) = \frac{\binom{4}{2}}{16} + \frac{\binom{4}{3}}{16} + \frac{1}{16} = \frac{11}{16}.$$

We conclude that $P(A|B) = 1/11$.

EXAMPLE 5.3. Let $\underline{n} = \{1, 2, \ldots, n\}$ and let $\pi : \underline{n} \to \underline{n}$ be a randomly selected permutation. Let A be the event that $\pi(1) > \pi(2)$. Let B be the event that $\pi(2) > \pi(3)$. What is $P(A|B)$? Are A and B independent events?

Clearly $P(A) = P(B) = 1/2$. Note that $A \cap B$ is the event that $\pi(1) > \pi(2) > \pi(3)$. Since there are $3! = 6$ orderings of 3 numbers, $P(A \cap B) = 1/6$. Thus $P(A|B) = P(A \cap B)/P(B) = 1/3$. Since $P(A) = 1/2 \neq 1/3$, we conclude that A and B are not independent.

It is relatively intuitive that the events of Example 3 are not independent. After all, if $\pi(2) > \pi(3)$, then $\pi(2)$ is "on the big side," so it will be harder for it to be smaller than $\pi(1)$. But be careful in applying this sort of reasoning. Intuition can easily lead us astray in probability theory, as the following example demonstrates.

EXAMPLE 5.4. During the 2016 Renn Fayre softball tournament, Professor Pavid Derkinson had a higher batting average than Professor Fim Jix. The same is true of their batting averages during the 2017 tournament. Does it follow that Derkinson's cumulative 2016–17 batting average is higher than Jix's?

Counterintuitively – but unsurprisingly given the setup – the answer is NO, not necessarily. Indeed, consider the following statistics.

		2016	2017	2016–17
Jix	hits	3	24	27
	at bats	10	60	70
	average	.300	.400	.386
Derkinson	hits	10	3	13
	at bats	30	5	35
	average	.333	.600	.371

We see that Derkinson has higher batting averages each season, but Jix has the higher cumulative batting average!

This counterintuitive phenomenon is pervasive and important enough to merit a name: *Simpson's paradox*. Note that there is no real paradox here, only something that goes against our intuition. In order to put a finer point on how and why Simpson's paradox arises, we turn to the Law of Total Probability.

THEOREM 5.5 (Law of Total Probability). *Let A and B be mutually exclusive events ($A \cap B = \varnothing$) such that $A \cup B = S$ and $P(A)P(B) > 0$. Then for any event C,*

$$P(C) = P(C|A)P(A) + P(C|B)P(B).$$

We can interpret this theorem as saying that the probability of C is the weighted average of its conditional probabilities. (Here $P(A)$ and $P(B)$ are the weights. Note that the hypotheses imply that $P(A) + P(B) = 1$, so this really makes sense as a weighted average.)

PROOF. Note that $A \cap C$ and $B \cap C$ are disjoint and $(A \cap C) \cup (B \cap C) = C$. Thus $P(C) = P(C \cap A) + P(C \cap B)$. Meanwhile,

$$\begin{aligned} P(C|A)P(A) + P(C|B)P(B) &= \frac{P(C \cap A)}{P(A)}P(A) + \frac{P(C \cap B)}{P(B)}P(B) \\ &= P(C \cap A) + P(C \cap B). \end{aligned}$$

We conclude that the two quantities are equal. □

In the case of Example 5, we get the following clearer picture of our softball heroes' batting averages. Let Hit_J be the event of Jix getting a hit in 2016 or 2017 and similarly define Hit_D to be

the event of Derkinson geting a hit in either season. Let J_{2016} denote Jix's at bats in 2016, and let J_{2017} denote his bats in 2017. Let D_{2016} denote Derkinson's at bats in 2016, similarly define D_{2017}. Then by Bayes' Theorem (moral exercise: check that the hypotheses hold!),

$$P(\text{Hit}_J) = P(\text{Hit}_J\,|J_{2016})P(J_{2016}) + P(\text{Hit}_J\,|J_{2017})P(J_{2017}), \text{ and}$$
$$P(\text{Hit}_D) = P(\text{Hit}_D\,|D_{2016})P(D_{2016}) + P(\text{Hit}_D\,|D_{2017})P(D_{2017}).$$

In the setup of Example 5, we know that all of the "J" conditional probabilities are smaller than their matching "D" conditional probabilities, but we have no control over how the "weights" $P(J_{2016})$, *etc.* compare. It turns out they can spoil our intuition and result in the "paradox" of $P(\text{Hit}_J) > P(\text{Hit}_D)$.

We conclude this lecture by considering how to generalize independence and Bayes' Theorem when there are more than two events. For independence, the right generalization is the maximally strong one.

DEFINITION 5.6. Events $A_1, \ldots, A_n$ are *independent* if for any nonempty set $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$,

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

We get the following generalization of Theorem 5.5 via a completely analogous proof. (Moral exercise: check the details.)

THEOREM 5.7. *Let $A_1, \ldots, A_n$ be events in the same sample space S such that $A_1 \cup \cdots \cup A_n = S$, $P(A_i) \neq 0$ for all i, and $A_i \cap A_j = \varnothing$ for all $i \neq j$. Let $C \subseteq S$ be any event. Then*

$$P(C) = P(C|A_1)P(A_1) + \cdots + P(C|A_n)P(A_n).$$

We conclude by giving a name to an easy algebraic trick with significant computational ramifications.

THEOREM 5.8 (Bayes' Law). *If $P(A), P(B) \neq 0$, then*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

PROOF. By the definition of conditional probability, we have $P(B|A) = P(B \cap A)/P(A)$, so the right-hand side of Bayes' Law becomes

$$\frac{P(B \cap A)}{P(B)} = \frac{P(A \cap B)}{P(B)} = P(A|B)$$

as desired. □

6. Day 27

PROBLEM 6.1 (The Monty Hall problem). A game show provides contestants with the opportunity to win a car. There are three doors labeled A, B, and C. Behind two of the doors are goats, and behind one of the doors is a car. For reasons not completely clear to your instructor, you hope to select the car instead of a goat. The game proceeds in the following fashion: First, you select a door. Next, the host reveals a goat behind one of the remaining doors. (Since there are two goats, there is at least one goat to reveal.) You are then given the chance to switch your guess. If your final guess is the door with the car behind it, you win the car. **Question:** Is it advantageous to switch your guess?

Here are some assumptions on the problem which should remove any ambiguity:

» The probability that the car is placed behind any one of the three doors is $1/3$.
» The host knows where the car is.

» If the contestant picks a door with a goat behind it at the beginning, the host opens the remaining door with a goat before giving the option to switch. If the contestant picks the door with the car behind it, the host opens any of the other doors with probability $1/2$.

Suppose that you initially pick door A and then let A, B, and C denote the events "the car is behind door A," "door B," and "door C," respectively. Let M_A, M_B, and M_C denote the events "the host opens door A," "door B," and "door C," respectively.

(a) What are $P(M_C|A)$, $P(M_C|B)$, and $P(M_C|C)$?
(b) What is $P(M_C)$? (Use the Law of Total Probability.)
(c) Suppose that the host opens door C revealing a goat. You should switch your guess to B if $P(B|M_C) > P(A|M_C)$. Compute these conditional probabilities (via Bayes' Law) and draw a conclusion.

PROBLEM 6.2. A student taking a true-false test always marks the correct answer when she knows it and decides true or false on the basis of flipping a fair coin when she does not know it. If the probability that she will know an answer is $3/5$, what is the probability that she knew the answer to a correctly marked question?

7. Expected value

In this lecture, we will study *random variables* and *expected value*. By the end of it, we should be able to precisely formulate and answer questions such as "How much can I expect to win if I play the lottery?" and "What is the expected number of fixed points for a random permutation?" Throughout, P is a probability distribution on a finite sample space S.

DEFINITION 7.1. A *random variable* is a function $X : S \to \mathbb{R}$.

In other words, a random variable is some way of assigning numbers to elements of a sample space. Note that we can add and multiply random variables X, Y on the same sample space, and we can also scale random variables by a real number. For $s \in S$ and $c \in \mathbb{R}$ these operations are given by the rules

$$(X + Y)(s) = X(s) + Y(s),$$
$$(XY)(s) = X(s)Y(s),$$
$$(cX)(s) = c \cdot (X(s)).$$

We can also assign an expected value (also called expectation, average value, or mean) to every random variable.

DEFINITION 7.2. Let $X : S \to \mathbb{R}$ be a random variable and let $X(S) = \{X(s) \mid s \in S\}$ denote the image of X. Then the number

$$E(X) := \sum_{y \in X(S)} y \cdot P(X = y)$$

is called the *expected value* of X on S. Here $P(X = y)$ is shorthand for the probability of the event $\{s \in S \mid X(s) = y\}$, *i.e.* the event that random variable X takes the value y.

In other words, $E(X)$ is the weighted average of the values X takes, with weights given by the probability that X takes the corresponding value.

EXAMPLE 7.3. A lottery offers \$1 tickets on which you choose six distinct numbers between 1 and 48, inclusive. The lottery announces winning numbers and if your ticket matches all the winning numbers (irrespective of order) you get \$1,000,000; otherwise you get nothing. Expected value allows us to at least partially answer the question "Should you play this lottery?"

Let S be the sample space of 6-element subsets of $\underline{48} = \{1, 2, \ldots, 48\}$. Define $X : S \to \mathbb{R}$ such that $X(s) = -1$ is s is not the winning ticket (because you've then lost your \$1 investment) and $X(s) = 999\,999$ if s is the winning ticket (the million dollar prize minus the ticket cost). Then $X(S) = \{-1, 999\,999\}$ and the expected value of X is

$$E(X) = -1 \cdot \frac{\binom{48}{6} - 1}{\binom{48}{6}} + 999\,999 \cdot \frac{1}{\binom{48}{6}} \approx -0.918.$$

This means that if you play this lottery many many times, then in the long run you can expect to lose about 92 cents each time you play, so it's not a good investment.

Expected value has an unexpected property: *linearity*. For those who have experience with linear algebra, this literally means that E, as a function from the $\mathbb{R}$-vector space of random variables to $\mathbb{R}$, is a linear transformation. If you don't speak that language yet, consider the following simply stated theorem as a definition of the term.

THEOREM 7.4. *Let $X, Y : S \to \mathbb{R}$ be random variables and let $c \in \mathbb{R}$. Then*

$$E(X + Y) = E(X) + E(Y)$$

and

$$E(cX) = cE(X).$$

Linearity of expected value is an extremely powerful tool. For the moment, we defer its proof and insetead use it to give a simple proof of the following remarkable fact.

THEOREM 7.5. *The expected value of the number of fixed points in a randomly selected permutation of $\underline{n} = \{1, 2, \ldots, n\}$ is 1.*

PROOF. Recall that a permutation π has i as a fixed point if $\pi(i) = i$. For $1 \leq i \leq n$ and π a permutation of $\underline{n}$, let $X_i(\pi) = 1$ if $\pi(i) = i$ and let $X_i(\pi) = 0$ otherwise. Define $X := X_1 + X_2 + \cdots + X_n$. Then $X(\pi)$ is equal to the number of fixed points of π and we are trying to find $E(X)$. By linearity, it suffices to find $E(X_i)$ for each i and then add up the values.

For a random permutation π of $\underline{n}$, $\pi(i)$ is equally likely to take any of the values in $\underline{n}$. Thus $P(X_i = 1) = 1/n$ and $P(X_i = 0) = (n-1)/n$. As such,

$$E(X_i) = 1 \cdot \frac{1}{n} + 0 \cdot \frac{n-1}{n} = \frac{1}{n}$$

for each $1 \leq i \leq n$. Thus

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1.$$

□

Note that Theorem 6 holds for any natural number n, so we say that the expected number of fixed points of a permutation of a finite set is 1.

7.1. Linearity of expected value. In this optional subsection, we'll look at another application of linearity of expectation, and then provide the promised proof of Theorem 7.4.

EXAMPLE 7.6. Consider the sample space $S = \underline{6} \times \underline{6}$ of two rolls of a fair 6-sided die. Define the random variable $X : S \to \mathbb{R}$ to be the sum of the two rolls. We will compute the expected value of X in two ways: first, via the definition of expectation, then via linearity of expectation.

The sum of two rolls is any integer between 2 and 12, inclusive, so $X(S) = \{2, 3, \ldots, 12\}$. We need to compute $P(X = 2)$, $P(X = 3)$, …, $P(X = 12)$.

We can only have $X = 2$ if both rolls take the value 1, so $P(X = 2) = 1/6^2 = 1/36$. We can get

$X = 3$ only with rolls $(1,2)$ and $(2,1)$, so $P(X = 3) = 2/36$. For $X = 4$ we have rolls $(1,3)$, $(2,2)$, $(3,1)$, so $P(X = 4) = 3/36$. For $X = 5$ we have rolls $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$, so $P(X = 5) = 4/36$. For $X = 6$ we have rolls $(1,5)$, $(2,4)$, $(3,3)$, $(4,2)$, $(5,1)$, so $P(X = 6) = 5/36$. For $X = 7$ we have rolls $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, $(6,1)$, so $P(X = 7) = 6/36$. For $X = 8$ (now things get interesting), we have rolls $(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, $(6,2)$, so $P(X = 8) = 5/36$. For $X = 9$ we have rolls $(3,6)$, $(4,5)$, $(5,4)$, $(6,3)$, so $P(X = 9) = 4/36$. For $X = 10$ we have rolls $(4,6)$, $(5,5)$, $(6,4)$, so $P(X = 10) = 3/36$. For $X = 11$ we have rolls $(5,6)$ and $(6,5)$, so $P(X = 11) = 2/36$. Finally, for $X = 12$ we have the single roll $(6,6)$ so $P(X = 12) = 1/36$. We conclude that

$$\begin{aligned} E(X) &= 2\frac{1}{36} + 3\frac{2}{36} + 4\frac{3}{36} + 5\frac{4}{36} + 6\frac{5}{36} + 7\frac{6}{36} + 8\frac{5}{36} + 9\frac{4}{36} + 10\frac{3}{36} + 11\frac{2}{36} + 12\frac{1}{36} \\ &= \frac{252}{36} \\ &= 7. \end{aligned}$$

Linearity provides a much less labor intensive way to compute the expected value of X. Define $X_1 : S \to \mathbb{R}$ to be the value of the first roll, and X_2 to be the value of the second role. Then $X = X_1 + X_2$, so $E(X) = E(X_1) + E(X_2)$. Since each roll is no different from the other, we have $E(X_1) = E(X_2)$, and thus $E(X) = 2E(X_1)$. Now it is quite easy to compute $E(X_1)$ since $P(X_1 = 1) = P(X_1 = 2) = \cdots = P(X_1 = 6) = 1/6$. Thus

$$\begin{aligned} E(X_1) &= 1\frac{1}{6} + 2\frac{1}{6} + \cdots + 6\frac{1}{6} \\ &= \frac{1 + 2 + \cdots + 6}{6} \\ &= \frac{6 \cdot 7/2}{6} \\ &= \frac{7}{2}. \end{aligned}$$

We conclude that $E(X) = 2 \cdot 7/2 = 7$.

We now proceed to the proof of Theorem 7.4 for which we will need the following equivalent formulation of expected value.

LEMMA 7.7. If $X : S \to \mathbb{R}$ is a random variable, then

$$E(X) = \sum_{s \in S} X(s)P(s).$$

(Here we are abusing notation and writing $P(s)$ for $P(\{s\})$.)

PROOF. For each $y \in X(S)$, let $X^{-1}y := \{s \in S \mid X(s) = y\}$. Then

$$\begin{aligned} \sum_{s \in S} X(s)P(s) &= \sum_{y \in X(S)} \sum_{s \in X^{-1}y} X(s)P(s) && \text{(grouping like terms)} \\ &= \sum_{y \in X(S)} \sum_{s \in X^{-1}y} yP(s) && \text{(since } X(s) = y \text{ for } s \in X^{-1}y\text{)} \\ &= \sum_{y \in X(S)} y \sum_{s \in X^{-1}y} P(s) && \text{(factoring).} \end{aligned}$$

It remains to show that $\sum_{s \in X^{-1}y} P(s) = P(X = y)$, but this follows from the axioms for a probability distribution since $\bigcup_{s \in X^{-1}y} \{s\}$ is a partition of the event $\{s \in S \mid X(s) = y\}$. □

PROOF OF THEOREM 7.4. Given the lemma, the proof is an exercise is tracing through definitions. We will prove the first statement and leave the second one as a moral exercise for the reader.

We have

$$\begin{aligned} E(X+Y) &= \sum_{s\in S}(X+Y)(s)P(s) && \text{(Lemma 3)} \\ &= \sum_{s\in S}X(s)P(s) + \sum_{s\in S}Y(s)P(s) && \text{(definition of } X+Y \text{ and distribution)} \\ &= E(X)+E(Y) && \text{(Lemma 3 twice),} \end{aligned}$$

as desired. □

8. Day 28

PROBLEM 8.1. The digits 1, 2, 3, 4 are randomly arranged into two two-digit numbers $\overline{AB}$ and $\overline{CD}$. In this problem you will ultimately determine the expected value of $\overline{AB}\cdot\overline{CD}$.

(a) If two of the digits 1, 2, 3, 4 are randomly selected (without replacement), what is their expected product?
(b) Write $\overline{AB}$ as a linear combination of the digits A and B. Similarly express $\overline{CD}$ in terms of C and D.
(c) Finally, use linearity of expectation and your answer to (a) to determine $E(\overline{AB}\cdot\overline{CD})$.

PROBLEM 8.2 (The coupon collector problem). Safeway is running a promotion in which they have produced n coupons and you randomly receive a coupon each time you check out. You passionately hope to one day collect all n coupons. What is the expected number of times T you'll have to check out at the store in order to collect all n? There's a very clever way to solve this problem with linearity of expectation!

(a) Label the coupons C_1, C_2, …, C_n. If $n = 4$, a successful collection of all 4 coupons might look like $C_2\ C_2\ C_4\ C_2\ C_1\ C_3$. Break the sequence into segments where a segment ends when you receive a new coupon. In the example sequence, the segments are C_2, $C_2\ C_4$, $C_2\ C_1$, C_3. Because it will make our lives easier and Kyle is a benevolent problem-writer, consider these the 0-th, 1-st, …, 3-rd segments (as opposed to 1-st through 4-th). Let X_k be the length of the k-th segment, and note that k ranges from 0 through $n-1$. In the example, $X_0 = 1$, $X_1 = 2$, $X_2 = 2$, and $X_3 = 1$. Express T, the total number of checkouts needed to collect all coupons, as a linear combination of the X_k.
(b) Compute p_k, the probability that you will collect a new coupon given that you have already collected k of them. After studying the geometric distribution in Lecture 5, we will learn that $E(X_k) = 1/p_k$. Compute this value.
(c) Use your answers to (a) and (b) to determine $E(T)$.
(d) Can you say anything about the asymptotic behavior of $E(T)$?

9. Bernoulli, binomial, indicator, and geometric random variables

Remember that a random variable $X : S \to \mathbb{R}$ assigns a value to each outcome in a sample space. Say we're running an experiment, and all we care about is whether it succeeds or not. We can model this with a **Bernoulli random variable** X, where $X = 1$ if the experiment is a success and $X = 0$ otherwise. In this case $P(X = 1)$ is usally denoted p and $P(X = 0)$ as $q = 1 - p$.

If we do a sequence of independent experiments, each of which results in success with probability p and failure with probability $q = 1 - p$, and we are interested in the number of successes we can model this with a **binomial random variable**.

EXAMPLE 9.1. We have a (possibly unfair) coin, which lands on heads with probability p and tails with probability q. If I flip the coin 3 times, what is the probability of getting exactly two heads?

Let X be the number of heads out of 3 flips. Then

$$P(X = 2) = p \cdot p \cdot q + p \cdot q \cdot p + q \cdot p \cdot p = \binom{3}{2} p^2 q.$$

This is why X is called a binomial random variable. If instead I flip the coin n times, the probability of getting exactly k heads is

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

Additionally, notice that

$$\sum_{k=0}^{n} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p + q)^n = 1$$

by the Binomial Coefficient Theorem, so all the probabilities sum to 1 as we expect.

To find the expected number of heads after n flips, we can make our lives easier by using the linearity trick. $X = I_1 + I_2 + \ldots + I_n$ where

$$I_j = \begin{cases} 1 & \text{if the coin is heads on the } j\text{th flip} \\ 0 & \text{otherwise} \end{cases}$$

These I_j are called **indicator random variables** because they indicate when a certain condition is met. Then for any j,

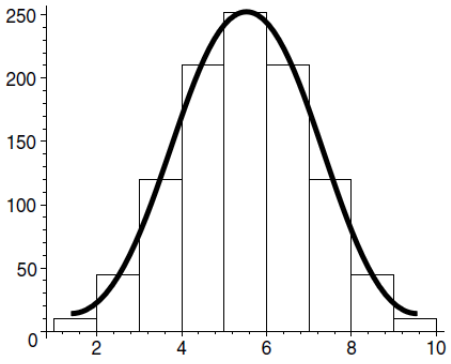
$$E[I_j] = 0 \cdot P(I_j = 0) + 1 \cdot P(I_j = 1) = p$$

so

$$E[X] = E[I_1] + E[I_2] + \ldots + E[I_n] = np.$$

If we graph the probabilities associated with a binomial random variable, they have a particular shape.

EXAMPLE 9.2. If $n = 10$ and $p = \frac{1}{2}$, then $P(X = k) = \binom{10}{k} \left(\frac{1}{2}\right)^{10}$.



Graph of $\binom{10}{k} = P(X = k) \cdot 2^{10}$

As n gets bigger, this approaches a bell curve, or Gaussian distribution. It is appropriate to approximate the probability distribution of a binomial random variable with a Gaussian distribution if n is large enough (usually when np and nq are both significantly larger than 10).

If we again run a series of independent experiments, but we are interested in the number of attempts needed to obtain the first success, we can model this with a **geometric random variable** X, where $X = k$ means that it takes k trials for the first success. Since succeeding for the first time on the kth try means failing on all tries up to $k - 1$, $P(X = k) = q^{k-1}p$. Do all these probabilities still sum to 1?

You may have seem the trick in 112

$$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r} \qquad \text{if } |r| < 1.$$

Notice that

$$\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} q^{k-1}p = p + qp + q^2p + \ldots = p(1 + q + q^2 + \ldots) = p\left(\frac{1}{1-q}\right) = \frac{p}{p} = 1$$

EXAMPLE 9.3. We have a fair twenty-sided die. What is the probability that I roll a critical hit (20 on the die) within 6 rolls?
This is $P(X \leq 6)$ where $p = 1/20$ and $q = 19/20$. Then

$$\begin{aligned} P(X \leq 6) &= P(X=1) + P(X=2) + \ldots + P(X=6) \\ &= \frac{1}{20} + \frac{19}{20}\cdot\frac{1}{20} + \left(\frac{19}{20}\right)^2\cdot\frac{1}{20} + \left(\frac{19}{20}\right)^3\cdot\frac{1}{20} + \left(\frac{19}{20}\right)^4\cdot\frac{1}{20} + \left(\frac{19}{20}\right)^5\cdot\frac{1}{20} \\ &\approx 0.265 \end{aligned}$$

What is the expected number of rolls before I roll a 20? Intuition says that if I have a $1/20$ chance, then I'll probably roll one every 20 rolls. Through a similar infinite series trick to the one above,

$$E[X] = \sum_{k=1}^{\infty} k\cdot P(X=k) = p + 2q\cdot p + 3q^2\cdot p + 4q^3\cdot + \ldots = p(1 + 2q + 3q^2 + 4q^3\cdot + \ldots) = p\left(\frac{1}{(1-q)^2}\right) = \frac{1}{p}$$

so in this case the math confirms our intuition.

10. Day 29

PROBLEM 10.1. With your group, roll a pair of dice twelve times. Record the first roll on which you roll doubles and also the total number of doubles that you roll and report these numbers to the instructor. What is the expected number of doubles in twelve rolls? How long should it take to roll doubles? How do these numbers compare with the class's statistics?

PROBLEM 10.2. An airline has sold 205 tickets for a flight that can hold 200 passengers. Each ticketed person, independently, has a 5% chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

PROBLEM 10.3. If the same airline consistently oversells the flight from Problem 2 at the same rate, how many flights until we expect more ticketed passengers to show up than there are seats.

CHAPTER 3

Number theory

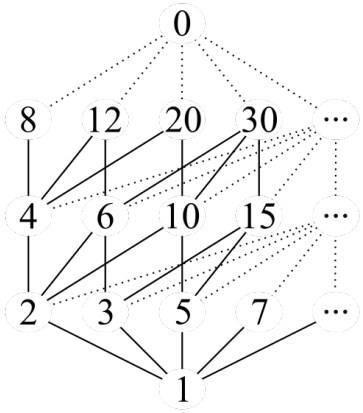
1. Day 30

For integers a, b, we say that a *divides* b when an integer m exists such that $b = am$; in this case we also say that b *is a multiple of* a and that a *is a divisor of* b.

QUESTION 1.1. When does $1 \mid b$? $-1 \mid b$? $a \mid 0$? $a \mid a$?

PROBLEM 1.2. Suppose that $a \mid b$ and $b \mid c$. Prove that $a \mid c$.

This produces a *partial order* on $\mathbb{N}$, visualized in the following diagram.



QUESTION 1.3. Where should you put 9 in the diagram?

PROBLEM 1.4. Prove that if $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ for all integers m, n.

A natural number $p > 1$ is *prime* if its only positive divisors are 1 and p. The fundamental theorem of arithmetic says that every positive integer is a product of primes, and that this factorization is unique up to reordering of the factors. For instance, $6 = 2 \cdot 3$, $1728 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^6 \cdot 3^3$ and $825 = 3 \cdot 5 \cdot 5 \cdot 11 = 3 \cdot 5^2 \cdot 11$. This probably seems like old hat, but not every number system has unique factorization! For instance, $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ supports addition and multiplication, but

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Number theorists are quite interested in objects like $\mathbb{Z}[\sqrt{-5}]$, but we will limit our study to $\mathbb{Z}$ where the fundamental theorem of arithmetic holds.

QUESTION 1.5. Where should the prime numbers go in the divisibility diagram?

PROBLEM 1.6. Prove that a positive integer n is prime if and only if n is not divisible by any prime p with $1 < p \leq \sqrt{n}$.

PROBLEM 1.7. Suppose that a positive integer n has prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$ with the p_i distinct primes. How many distinct positive integers are divisors of n?

PROBLEM 1.8. The book's proof does a fine job of guaranteeing that prime factorizations of integers are unique, but it elides the proof that prime factorization *exist*. Give an inductive proof that every positive integer has a prime factorization.

2. Day 31

The key takeaways from §6.4 are that there are infinitely many prime numbers, and that the prime counting function $\pi(n) = |\{p \in \mathbb{N} \text{ prime} \mid p \leq n\}|$ grows like $n/\log n$. (Here we are using $\log$ for the natural logarithm function.) The first of these results is generally attributed to Euclid, c. 300B.C.E. Let's look at another proof due to Filip Saidak from 2005. In order to get it off the ground, prove the following result.

PROBLEM 2.1. Let n be a positive integer. Prove that n and $n+1$ share no common divisors greater than 1.

PROOF THAT THERE ARE INFINITELY MANY PRIME NUMBERS. Let $n > 1$ be a positive integer. As we have just proven, n and $n+1$ share no common divisors greater than 1. Hence the number $N_2 = n(n+1)$ must have at least two distinct prime factors. Similarly, N_2 and $N_2 + 1$ share no common divisors greater than 1, and thus $N_3 = N_2(N_2+1)$ must have at least 3 distinct prime factors. We recursively define $N_k = N_{k-1}(N_{k-1}+1)$ for $k > 2$ and observe inductively that N_k has at least k distinct prime factors. □

Note that N_k has at least k distinct prime factors, each of which is necessarily smaller than N_k. It follows that $\pi(N_k) \geq k$.

QUESTION 2.2. Compute N_k for $2 \leq k \leq 5$. Is this a very effective bound on the prime counting function?

The vaunted Prime Number Theorem (PNT) says that

$$\pi(n) \sim \frac{n}{\log n},$$

which means that

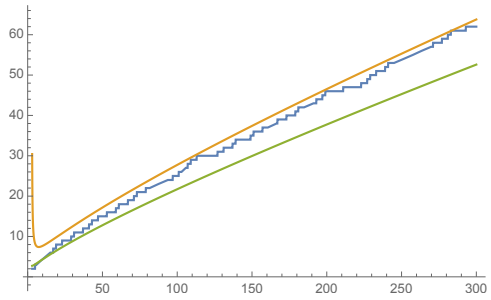
$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = \lim_{n\to\infty} \frac{\pi(n)\log n}{n} = 1.$$

The proof is very difficult and beyond the scope of this course, but we will still happily use the result.

PROBLEM 2.3. Show that $\lim_{n\to\infty} \pi(n)/n = 0$ and use this to show that for any $a \in \mathbb{R}$,

$$\pi(n) \sim \frac{n}{\log(n) - a}.$$

It turns out that $a = 1$ gives the best approximation to $\pi(n)$. In the below plot, the curve on top is the graph of $n/(\log(n) - 1$, the middle curve is the graph of $\pi(n)$, and the bottom curve is the graph of $n/\log n$.



3. Day 32

PROBLEM 3.1. As an intrepid wagon wheel painter living in the Olde West, you strive to bring the highest quality, most engaging, non-monochromatic spoke paintings to your customers. You offer wagon wheels with p spokes, where p is a prime integer, painted in up to a colors, where $1 \leq a \leq p-1$.

(a) As part of your preparation for painting, you have nailed a wagon wheel to the wall so that it can't rotate. In how many ways can you paint its spokes, assuming that each spoke gets a single color but at least two of the spokes are different colors?

(b) When you take the wheel off of the wall and fix it to an axle, you remember that it will rotate, and that your demanding customers will not accept rotated spoke paintings as genuinely different. As you turn this particular wheel around, you notice something remarkable: all of the rotations by multiples of $2\pi/p$ result in distinct colorings in the wheel-nailed-to-wall sense of unique, despite the fact that there are multiple spokes of the same color (since $a < p$). Is this a special property of your particular spoke painting, or is it true of all possible non-monochromatic paintings with a colors?

(c) Use your work in (b) to determine the total number wagon wheel paintings which your customers will accept as genuinely different. What can you deduce from the fact that this number is an integer?

PROBLEM 3.2. How many 6-spoke wheels can you paint non-monochromatically with up to a colors for $a = 2, 3, 4, 5$?

4. Day 33

The *greatest common divisor* $d = \gcd(a, b)$ of integers a, b is the largest positive integer such that $d \mid a$ and $d \mid b$. We say that a and b are *relatively prime* when they share no divisors larger than 1, and this is equivalent to $\gcd(a, b) = 1$.

PROBLEM 4.1. Draw a divisor diagram for 84 and 105. Where does the $\gcd$ appear in partially ordered set of divisors?

If we know the prime factorizations of a and b, this number is easy to determine. Let $\{p_1, p_2, \ldots, p_k\}$ be the set of distinct prime divisors of a and b. Then we may write

$$a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$
$$b = p_1^{b_1} p_2^{b_2} \cdots p_k^{a_k}$$

for nonnegative integers a_i, b_i and

$$\gcd(a, b) = p_1^{\min\{a_1, b_1\}} p_2^{\min\{a_2, b_2\}} \cdots p_k^{\min\{a_k, b_k\}}.$$

It is frequently the case, though, that we do not have access to the prime factorizations of integers. In this case, the *Euclidean algorithm* allows us to determine the greatest common divisor. Let's execute the algorithm with $a = 81$, $b = 57$:

$$81 = 1 \cdot 57 + 24$$
$$57 = 2 \cdot 24 + 9$$
$$24 = 2 \cdot 9 + 6$$
$$9 = 1 \cdot 6 + 3$$
$$6 = 2 \cdot 3 + 0.$$

We conclude that the final nonzero remainder, 3, is the $\gcd$ of 81 and 57. Indeed, $81 = 3^4$ and $57 = 3 \cdot 19$, so this agrees with our first method for determining $\gcd$'s.

The Euclidean algorithm can be described formally as follows:

1. Assume $a > b$ are integers (if $a < b$, swap them).
2. Perform long division to express to express $a = qb + r$ where $0 \leq r \leq b - 1$.
3. Replace a with b and b with r.
4. If $r \neq 0$, return to step 2; else
5. if $r = 0$, conclude that the final nonzero remainder is $\gcd(a, b)$.

A generic run of the algorithm then looks like

$$\begin{aligned} a &= q_0 b + r_1 \\ b &= q_1 r_1 + r_2 \\ r_1 &= q_2 r_2 + r_3 \\ r_2 &= q_3 r_3 + r_4 \\ &\vdots \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n \\ r_{n-1} &= q_n r_n + 0 \end{aligned}$$

where $1 \leq r_k \leq r_{k-1}$ and we conclude that $r_n = \gcd(a, b)$ (since $r_{n+1} = 0$).

PROBLEM 4.2. Why does the Euclidean algorithm work? Start at the end of the algorithm and check that $r_n \mid r_{n-1}$, then inductively check that $r_n \mid r_k$ for $-1 \leq k \leq n$ where we write $r_0 = b$ and $r_{-1} = a$ for notational convenience. Conclude that r_n divides a and b. Use a similar argument starting at the beginning of the algorithm to show that $\gcd(a, b)$ divides r_k for $-1 \leq k \leq n$. Why does this prove that the algorithm produces the $\gcd$.

PROBLEM 4.3. The Euclidean algorithm gives us a way to dissect a rectangle with integer sides into squares. Run the Euclidean algorithm to find $\gcd(23, 13)$. Interpret the first step ($23 = 1 \cdot 13 + 10$) as telling you that $q_0 = 1$-many 10×10 squares fit inside a 23×13 rectangle. Figure out what instructions the rest of the algorithm is giving you and draw a corresponding picture. At the end, your 23×13 rectangle should be partitioned into squares! What is special about this procedure if you start with consecutive Fibonacci numbers $a = F_{n+1}$, $b = F_n$?

PROBLEM 4.4. Run the Euclidean algorithm when $a = 45$, $b = 16$. How is it related to the expression

$$\frac{45}{16} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}}?$$

Come up with a general procedure by which the Euclidean algorithm produces *continued fraction* expressions for rational numbers of the form

$$\frac{a}{b} = x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4 + \cdots}}}$$

where the x_i are integers.

5. Day 34

The book says that integers a and b are congruent modulo another integer m (denoted $a \equiv b \pmod{m}$) if a and b have the same remainder upon division by m. In your homework, you will prove that this is equivalent to $m \mid a - b$, and you should assume this result for the rest of today's class.

QUESTION 5.1. When is $a \equiv b \pmod{2}$? $a \equiv b \pmod{1}$? $a \equiv b \pmod{0}$?

PROBLEM 5.2. Prove that $\equiv \pmod{m}$ is an equivalence relation on $\mathbb{Z}$. What are the associated equivalence classes? How many equivalence classes are there?

When considering the equivalence relation $\equiv \pmod{m}$ on $\mathbb{Z}$, we write $\overline{a}$ for the equivalence class of a. (We elide m from the notation; it should be clear from context.) We call $\overline{a}$ the congruence class of a modulo m. We write $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/(\equiv \pmod{m})$ for the set of congruence classes modulo m.

PROBLEM 5.3. Define addition and multiplication of equivalence classes in $\mathbb{Z}/m\mathbb{Z}$. Show that for every $\overline{a} \in \mathbb{Z}/m\mathbb{Z}$ there exists $\overline{b} \in \mathbb{Z}/m\mathbb{Z}$ such that $\overline{a} + \overline{b} = \overline{0}$.

Let's now shift gear and discuss the *dynamics* of addition in $\mathbb{Z}/m\mathbb{Z}$. Fix $\overline{a} \in \mathbb{Z}/m\mathbb{Z}$. Make a directed graph[1] $G(\overline{a}, m)$ with vertex set $\mathbb{Z}/m\mathbb{Z}$ such that $(\overline{b}, \overline{c})$ is an edge if and only if $\overline{c} = \overline{b} + \overline{a}$.

PROBLEM 5.4. Draw $G(\overline{a}, m)$ for a germane collection of $\overline{a}$ and m.

PROBLEM 5.5. Make a conjecture regarding the shape of $G(\overline{a}, m)$. Prove it.

6. Day 35

In §6.8 you learned that there are commutative, associative operations $+, \cdot$ on $\mathbb{Z}/n\mathbb{Z}$ and that $+$ admits an inverse $-$ such that $\overline{a} - \overline{a} = \overline{0}$. When n is prime, everything in $\mathbb{Z}/n\mathbb{Z}^\times = \mathbb{Z}/n\mathbb{Z} \smallsetminus \{\overline{0}\}$ admits a multiplicative inverse as well, *i.e.*, for each $\overline{a} \in \mathbb{Z}/n\mathbb{Z}^\times$, there exists $\overline{a}^{-1} \in \mathbb{Z}/n\mathbb{Z}^\times$ such that $\overline{a} \cdot \overline{a}^{-1} = \overline{1}$. We sometimes write $\overline{1}/\overline{a}$ for $\overline{a}^{-1}$ and $\overline{a}/\overline{b}$ for $\overline{a}\overline{b}^{-1}$.

PROBLEM 6.1. Our previous version of Fermat's little theorem said that if p was prime and $1 \leq a \leq p-1$, then $p \mid a^p - a$. Of course, $p \mid 0 = 0^p - 0$, so this holds for $0 \leq a \leq p-1$ as well.

(a) Check that this is equivalent to $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{Z}$.
(b) Suppose $a \not\equiv 0 \pmod{p}$. Prove that $a^{p-1} \equiv 1 \pmod{p}$.
(c) For $p > 2$, what are the possible values of $a^{(p-1)/2} \bmod p$? (Note that $p-1$ is even when $p > 2$, so $(p-1)/2$ makes sense.)
(d) For $a \in \mathbb{Z}$ such that $a \not\equiv 0 \pmod{p}$, define $o_p(a)$ (the *order of a modulo p*) to be the smallest positive integer such that $a^{o_p(a)} \equiv 1 \pmod{p}$. Since $a^{p-1} \equiv 1 \pmod{p}$, we know that $1 \leq o_p(a) \leq p-1$. Prove that $o_p(a) \mid p-1$.
(e*) Prove that there exists $a \in \mathbb{Z}$ such that $o_p(a) = p-1$.
(f) Assume (e*) (which is a challenge problem you can try outside of class) and take $a \in \mathbb{Z}$ such that $o_p(a) = p-1$. Show that each a^n, $1 \leq n \leq p-1$, is in a distinct congruence class modulo p and thus the values of a^n cycle through all the nonzero congruence classes mod p with period $p-1$.[2]

[1] The edges in a directed graph have a source and target, indicated by an arrow. Thus the edges in a directed graph are encoded by ordered pairs of vertices, with first entry the source, and second entry the target.

[2] An algebraist would say that $\mathbb{Z}/p\mathbb{Z}^\times$ is a cyclic group of order $p-1$.

PROBLEM 6.2. Make a multiplication table for $\mathbb{Z}/7\mathbb{Z}^\times$. Select a congruence class and circle all its occurrences in the table. Observe that this is a solution to the non-capturing rooks problem on a 6×6 chessboard. Does it work for other congruence classes? For $\mathbb{Z}/p\mathbb{Z}^\times$ and $(p-1) \times (p-1)$ chessboards in general? Why?

PROBLEM 6.3. How many squares are there mod p? *i.e.*, how large is $\{\overline{x}^2 \mid \overline{x} \in \mathbb{Z}/p\mathbb{Z}^\times\}$? What is the probability that $x^2 \equiv a \pmod{p}$ will have a solution? Suppose $x^2 \equiv a \pmod{p}$ has a solution; how many solutions does it have? In the diagonal of the multiplication table for $\mathbb{Z}/p\mathbb{Z}^\times$, why does $\overline{1}$ always and only appear in the top left and bottom right corner?

PROBLEM 6.4. Your vitamin regimen requires you to take *Doctor Snoggleswarf's Health Elixir* ® every five days. You take the first dose in the bottle on a Sunday and the final dose on a Thursday. You're not sure how many doses you took, but you know that there are at least 50 doses in a bottle. What is the minimum number of doses you took?

7. Day 36

Suppose $n = p_1^{a_1} \cdots p_k^{a_k}$ for positive integers a_i and distinct primes p_i. Recall that $\phi(n)$ is the number of positive integers smaller than n and relatively prime to n. We claim that

$$\phi(n) = n(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_k).$$

To prove this, we count the number of positive integers which are at most n and are not relatively prime to n. This is the case if and only if one of the p_i divides n. Of course, there are n/p_i positive integers $\leq n$ and divisible by p_i, so it is tempting to guess that $\phi(n) = n-(n/p_1+n/p_2+\cdots+n/p_k)$, but inclusion-exclusion tells us we need to be more careful with numbers which are divisible by multiple primes. The correct formula is

$$\phi(n) = n - \sum_{1 \leq i \leq k} \frac{n}{p_i} + \sum_{1 \leq i_1 < i_2 \leq k} \frac{n}{p_{i_1} p_{i_2}} - \sum_{1 \leq i_1 < i_2 < i_3 \leq k} \frac{n}{p_{i_1} p_{i_2} p_{i_3}} + \cdots \pm \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_k}}$$

where the signs alternate and the final sign is $+$ if k is even and $-$ if k is odd. Factoring out an n and thinking deeply about the distributive law, we see that this is the same as

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

What a remarkable formula! For instance, if $n = 6160 = 2^4 \cdot 5 \cdot 7 \cdot 11$, then

$$\phi(6160) = 6160(1 - 1/2)(1 - 1/5)(1 - 1/7)(1 - 1/11) = 1920.$$

Also note that there is a probabilistic interpretation of this formula. The probability that an integer between 1 and n is relatively prime to n is

$$\frac{\phi(n)}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

Fascinatingly, the probability only depends on the primes dividing n, and it suggests an alternate proof of our formula.

PROBLEM 7.1. Let $\underline{n}$ be our sample space with uniform distribution. Define the event ND_i to be the set of $r \in \underline{n}$ such that $p_i \nmid r$.

(a) What is $P(ND_i)$?
(b) Let RP be the collection of $r \in \underline{n}$ which are relatively prime to n. Check that $RP = ND_1 \cap ND_2 \cap \cdots \cap ND_k$.

(c) Argue that the events ND_i are independent and thus $P(RP) = P(ND_1)\cdots P(ND_k)$. Note that this is equivalent to the above formula for $\phi(n)$.

8. Sunzi's Theorem

The Chinese mathematician Sunzi Suanjing considered the following problem in the 3-rd century C.E. A general arrays his soldiers on the parade grounds. He first organizes them into columns of 3, but there are only 2 soldiers in the final column. He then organizes them into columns of 5, but there are only 3 soldiers in the final column. Finally, he organizes them into columns of 7, and again there are only 2 soldiers in the final column. How many soldiers does the general command?

Using the language of congruences, we can phrase the general's observations as

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 2 \pmod 7. \end{aligned}$$

What (if any) integers x simultaneously satisfy these congruences?

Let us begin by solving the first two congruences, $x \equiv 2 \pmod 3 \equiv 3 \pmod 5$. By guess-and-check, we quickly see that $x = 8$ is a solution. In fact, if $x \equiv 8 \pmod{15}$, we solve both congruences. Indeed, such x are equal to $15k+8$ for some $k \in \mathbb{Z}$, and $15 \equiv 0$ modulo both 3 and 5.

We now need to solve the congruences $x \equiv 8 \pmod{15} \equiv 2 \pmod 7$. A little thought reveals that $x = 23$ works, and the same logic as before shows that $x \equiv 23 \pmod{105}$ gives all solutions (because $105 = 15 \cdot 7$).

This brief exploration indicates the following theorem and its proof.

THEOREM 8.1 (Sunzi's Theorem [née Chinese Remainder Theorem]). *Suppose $N = n_1 n_2 \cdots n_k$ and that the n_i are pairwise relatively prime integers (so $\gcd(n_i, n_j) = 1$ for $i \neq j$). Then for any integers $a_1, \ldots, a_k$ the system of congruences*

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

has precisely one solution $x = x_0$ with $0 \leq x_0 < N$ and all solutions are of the form $x \equiv x_0 \pmod N$.

PROOF. We proceed by induction on k. If $k = 1$, then we may take x to be the remainder of a_1 divided by n_1 and clearly all solutions are of the form $x + n_1 r = x + Nr$, $r \in \mathbb{Z}$.

Fix $s \geq 1$ and suppose that all such systems with $k = s$ terms have solutions as described. Now consider a system of $s+1$ congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_s \pmod{n_s} \\ x &\equiv a_{s+1} \pmod{n_{s+1}}. \end{aligned}$$

where the n_i are pairwise relatively prime. Let us first endeavor to solve the first two congruences. Since n_1 and n_2 are relatively prime, there are integers m_1 and m_2 such that $1 = m_1 n_1 + m_2 n_2$. Construct the number $a_{1,2} = a_2 m_1 n_1 + a_1 m_2 n_2$. Since $m_1 n_1 = 1 - m_2 n_2$, we have $a_{1,2} = a_2(1 -$

$m_2n_2) + a_1m_2n_2 = a_2 + n_2(a_1m_2 - a_2m_2)$. Reducing mod n_2, we get $a_{1,2} \equiv a_2 \pmod{n_2}$. If we begin with the substitution $m_2n_2 = 1 - m_1n_1$, we similarly get $a_{1,2} \equiv a_1 \pmod{n_1}$. Thus $a_{1,2}$ is a simultaneous solution of the first two congruences. We get all such solutions by considering $x \equiv a_{1,2} \pmod{n_1n_2}$. (The diligent reader should check this.) Thus we can solve the original $s+1$ congruences by solving the system

$$\begin{aligned} x &\equiv a_{1,2} \pmod{n_1n_2} \\ x &\equiv a_3 \pmod{n_3} \\ &\vdots \\ x &\equiv a_{s+1} \pmod{n_{s+1}} \end{aligned}$$

with only s congruences. Note that all the moduli are relatively prime, so we may invoke the inductive hypothesis, and we are done. □

This method of proof is constructive, in that it provides us with a method via which we can solve our system of congruences. By repeated application of the extended Euclidean algorithm, we can eliminate congruences one at a time until we get to a final congruence $x \equiv a_{1,2,\ldots,k} \pmod{N}$, where $a_{1,2,\ldots,k}$ is our solution.

In practice, this is not the fastest way to find a solution. (It requires $k-1$ applications of the extended Euclidean algorithm.) Instead, suppose that n_k is the largest of the moduli. There are $N/n_k = n_1n_2\cdots n_{k-1}$ numbers x such that $0 \le x < N$ and $x \equiv a_k \pmod{n_k}$. If N/n_k is relatively small, we (or a computer) can simply check if each of these numbers satisfies all k congruences.

As an example, consider the system of congruences $x \equiv 0 \pmod 2 \equiv 1 \pmod 3 \equiv 2 \pmod 5 \equiv 3 \pmod 7$. The solutions to $x \equiv 3 \pmod 7$ with $0 \le x < 2\cdot 3\cdot 5\cdot 7 = 210$ are $x = 3, 10, 17, \ldots, 206$. Eliminating odd x we are left with $x = 10, 24, 38, 52, 66, 80, 94, 108, 122, 136, 150, 164, 178, 192, 206$ as possible solutions. It is easy to see that only $x = 52, 122, 192$ are congruent to $2 \pmod 5$, and then that only $x = 52$ is $1 \pmod 3$. We conclude that the only solutions to this system of congruences are integers $x \equiv 52 \pmod{210}$.

There is a direct way to construct solutions as well. Let $N_i = N/n_i$ for $i = 1, \ldots, k$. Observe that N_i and n_i are relatively prime, so we can find M_i and m_i such that

$$1 = M_iN_i + m_in_i.$$

The reader may check that

$$x = \sum_{i=1}^{k} a_iM_iN_i$$

is a solution to the system of congruences, and thus all solutions are of the form

$$x \equiv \sum_{i=1}^{k} a_iM_iN_i \pmod{N}.$$

This recipe gives us a function

$$\begin{aligned} f\colon \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z} &\longrightarrow \mathbb{Z}/N\mathbb{Z} \\ (a_1, a_2, \ldots, a_k) &\longmapsto \sum_{i=1}^{k} a_iM_iN_i \end{aligned}$$

(We have engaged in the standard subterfuge of conflating integers and their congruence classes.) There is another natural function $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ sending x to the k-tuple consisting of the reductions of x modulo each n_i. The interested reader may check that these

functions are inverse to each other, and thus these sets are in bijection. In fact, these assignment also respect addition and thus are *isomorphisms of abelian groups*, a topic one can explore more fully in Math 332!

PROBLEM 8.2. Find all solutions to the system of congruences

$$x \equiv 2 \pmod{11}$$
$$x \equiv 3 \pmod{12}$$
$$x \equiv 4 \pmod{13}.$$

PROBLEM 8.3. Does Sunzi's theorem still hold if we drop the requirement that the n_i are relatively prime? Prove your assertion or provide a counterexample.

9. Day 37

QUESTION 9.1. Solve the system of congruences

$$2x \equiv 5 \pmod{7}$$
$$3x \equiv 4 \pmod{8}.$$

PROBLEM 9.2. What is the remainder when you divide 135^3 by 1728? (*Hint*: $1728 = 64 \cdot 27$.)

Recall that the Fermat-Euler Theorem is a generalization of Fermat's Little Theorem which states that

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

when $\gcd(a, n) = 1$. We will prove a special case of this theorem in which n is the product of k distinct primes, $n = p_1 p_2 \cdots p_k$. In this case, $\phi(n) = (p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$. Let $q_i = \phi(n)/(p_i - 1)$ for $i = 1, 2, \ldots, k$. Then

$$a^{\phi(n)} = (a^{p_i - 1})^{q_i} \equiv 1^{q_i} \equiv 1 \pmod{p_i}$$

for all i. We see then that $x = a^{\phi(n)}$ is a simultaneous solution of the congruences

$$x \equiv 1 \pmod{p_1},\ x \equiv 1 \pmod{p_2},\ \ldots,\ x \equiv 1 \pmod{p_k}.$$

But $x = 1$ is another solution! By Sunzi's theorem, it follows that $a^{\phi(n)} \equiv 1 \pmod{n}$. □

PROBLEM 9.3. How can the above argument be extended to the case in which $n = p_1^{a_1} \cdots p_k^{a_k}$ where the p_i are distinct primes and $a_i \geq 1$?

CHAPTER 4

Solutions

1. Day 1

SOLUTION TO QUESTION 1.1. Placing n rooks along the diagonal of an $n \times n$ chessboard exhibits a non-attacking configuration. We can enumerate all examples by placing rook 1 in any of the n positions in the first column, placing rook 2 in any of the $n-1$ positions in the second column not attackable by the first rook, placing rook 3 in any of the $n-2$ positions in the third column not attackable by the first two rooks, *etc*. For rook k, there are $n-k+1$ possibilities in the k-th column, and for the final rook there are $n-n+1=1$ possible placements in the n-th column. In total, there are $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ non-attacking configurations of n rooks on an $n \times n$ chessboard. □

2. Day 2

SOLUTION TO QUESTION 2.5. Begin by artificially labeling the letters

$$M_1, I_1, S_1, S_2, I_2, S_3, S_4, I_3, P_1, P_2, I_4$$

and noting that there are 11 letters total. There are 11! ways to arrange the labeled letters (11 choices for the first letter, 10 for the second, *etc*). But this overcounts: given a particular word of labeled letters, we can rearrange the I's in 4! ways, rearrange the S's in 4! ways, and rearrange the P's in 2! ways and still get the same word of unlabled letters. Thus there are

$$\frac{11!}{4!4!2!} = 34,560$$

ways to rearrange the letters in MISSISSIPPI. □

SOLUTION TO QUESTION 1.2. By an $n \times k$ grid, let's assume we mean ordered pairs of integers (a, b) where $0 \le a \le n$ and $0 \le b \le k$. Our aim is to go from $(0,0)$ to (n,k) without leaving the $n \times k$ grid and while only taking unit steps right or up.

First note that we have to take $n+k$ total steps to achieve our goal. Furthermore, exactly n of those steps can go right, and exactly k of those steps can go up (otherwise we don't get far enough or we leave the grid). Thus we can count the number of monotonic paths by counting the number of "words" with n R's (for right) and k U's (for up).

As a first approximation, we can label the R's $R_1, R_2, \ldots, R_n$ and the U's $U_1, U_2, \ldots, U_k$. There are $(n+k)!$ ways to order these distinguishable letters. But this is an overcount! The words $R_1R_2U_1R_3U_2$ and $R_3R_2U_2R_1U_1$ both correspond to $RRURU$; any re-ordering of the R's and any reordering of the U's gives the same word. Thus there are

$$\frac{(n+k)!}{n!k!}$$

monotonic paths. □

REMARK 2.1. What does this have to do with the "Galton board" of Figure 2? Label the top center peg $(0,0)$. As the ball bounces down, it bounces either right or left (corresponding to R or

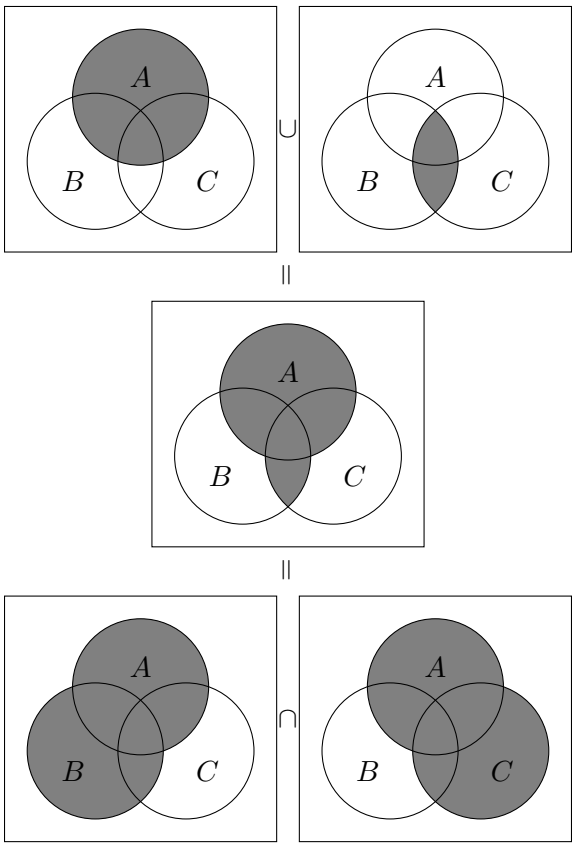


FIGURE 1. A graphical representation of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

U in a monotonic path). We have counted the total number of ways the ball can bounce so as to land in the trough labeled (n, k). We'll have more to say about this and the so-called binomial distribution later.

REMARK 2.2. Later, we will identify the number $(n + k)!/(n!k!)$ as the *binomial coefficient* $\binom{n+k}{n} = \binom{n+k}{k}$, a quantity some of you may know something about already. For now, just keep this fact in mind.

3. Day 3

SOLUTION TO PROBLEM 3.1. Yes! See Figure 1 for a graphical representation of this fact.

For a more formal proof, let $X = A \cup (B \cap C)$ and let $Y = (A \cup B) \cap (A \cup C)$. As is typical, we prove that $X = Y$ by showing that $X \subseteq Y$ and $Y \subseteq X$.

$X \subseteq Y$: Suppose $x \in X$, which means that $x \in A$ or $x \in B \cap C$, *i.e.*, $x \in A$ or ($x \in B$ and $x \in C$). If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in Y$; if $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in Y$. We have thus seen that $x \in X$ implies $x \in Y$, so $X \subseteq Y$.

$Y \subseteq X$: Suppose $y \in Y$, so $y \in A \cup B$ and $y \in A \cup C$. For the first condition to hold, $y \in A$ or $y \in B$. Equivalently, $y \in A$ or $y \in B \setminus A$. (Do you see why?) If $y \in A$, then $y \in X = A \cup (B \cap C)$; if $y \in B \setminus A$, then since $y \in A \cup C$, it must be in C (since it's not in A). Thus when $y \in B \setminus A$, $y \in B$ and $y \in C$, *i.e.*, $y \in B \cap C$, whence $y \in X$. No matter what, whenever $y \in Y$, y is also in X, so $Y \subseteq X$.

We have just seen that $X \subseteq Y$ and $Y \subseteq X$, so $X = Y$. □

SOLUTION TO QUESTION 3.2. The set $A \times C$ consists of pairings (a, c) where $a \in A$ and $c \in C$, so $A \times C$ consists of all possible adult-child pairings. There are 30 adults, each of which can be paired with any of the 50 children, so there are $30 \cdot 50 = 1,500$ possible pairings. □

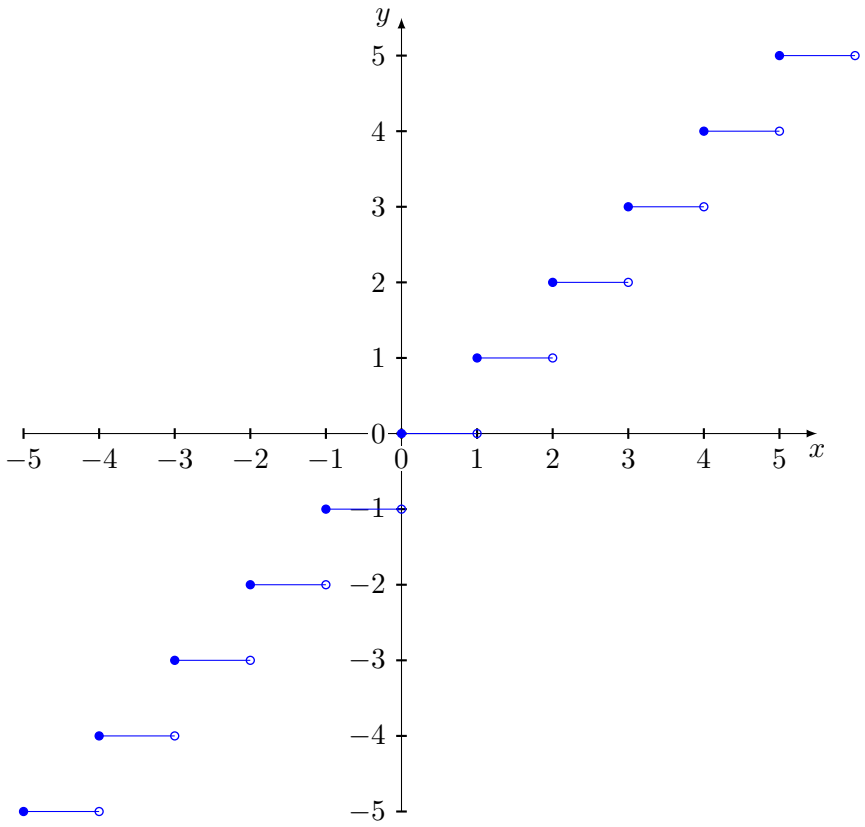
SOLUTION TO PROBLEM 3.3. We claim that $|A \times B| = |A||B|$. Indeed, there are $|A|$ ways to fill the first entry, and $|B|$ ways to fill the second. □

SOLUTION TO PROBLEM 3.4. (a) This is a function: each element of $\{1, 2, 3\}$ appears in the first coordinate precisely once, and the second entries are all elements of $\{a, b, c, d\}$.
(b) This is not a function since no term of the form $(1, y)$ appears in the set.
(c) This is not a function since 2 appears twice in the first coordinate.
(d) This is a function. (It's fine for elements of the codomain to be repeated. This particular function is called the *constant function with value* a.)

□

4. Day 4

SOLUTION TO PROBLEM 5.1. A graph of $\lfloor\ \rfloor$ looks like this:



The image of $\lfloor\ \rfloor$ is exactly the set of integers, $\mathbb{Z}$. Indeed, for $n \in \mathbb{Z}$, $\lfloor n \rfloor = n$, so $\mathbb{Z} \subseteq \text{im}\lfloor\ \rfloor$. By its definition, $\lfloor x \rfloor \in \mathbb{Z}$ for all $x \in \mathbb{R}$, so $\text{im}\lfloor\ \rfloor \subseteq \mathbb{Z}$ as well, hence $\text{im}\lfloor\ \rfloor = \mathbb{Z}$.

Since the image of the floor function is not its entire codomain, $\mathbb{R}$, it is not surjective; furthermore, the floor function is not injective since, for instance, $\lfloor 0 \rfloor = 0 = \lfloor 1/2 \rfloor$. □

SOLUTION TO PROBLEM 5.2. We first show that f is injective. Suppose that $f(n) = f(m)$. If n and m are both even, then we know $n/2 = m/2$, and multiplying by 2 we conclude that $n = m$. If n and m are both odd, then we know $(-1-n)/2 = (-1-m)/2$; multiplying by 2, adding 1, and then multiplying by -1, we get $n = m$. If n is even and m is odd, then by definition $f(n) = n/2 \geq 0$ and $f(m) = (-1-m)/2 < 0$, a contradiction. If n is odd and m is even, we similarly get $f(n) < 0$ and $f(m) \geq 0$, a contradiction. We conclude that whenever $f(n) = f(m)$, in fact $n = m$, so f is injective.

We now show that f is surjective, concluding our proof of bijectivity. If a is a nonnegative integer, then $2a$ is an even natural number and $f(2a) = (2a)/2 = a$. If a is a negative integer, then $-1-2a$ is an odd natural number, and $f(-1-2a) = (-1-(-1-2a))/2 = a$. We conclude that $\operatorname{im} f = \mathbb{Z}$, so f is surjective. □

SOLUTION TO PROBLEM 5.3. If $f : A \to B$ is injective, then $|A| \leq |B|$. If $f : A \to B$ is surjective, then $|A| \geq |B|$. □

SOLUTION TO PROBLEM 5.4. We claim that there are $|B|^{|A|}$ such functions. Indeed, for each element of the domain, we can assign any of $|B|$ different potential values. Since there are $|A|$ elements of A, the count amounts to taking the $|A|$-fold iterated product of $|B|$ with itself, *i.e.*, $|B|^{|A|}$. □

REMARK 4.1. The set $F(A, B)$ is often denoted B^A. With this notation, we have just shown that

$$\left|B^A\right| = |B|^{|A|}.$$

SOLUTION TO PROBLEM 5.5. By definition, $f(\varnothing) = \{f(a) \mid a \in \varnothing\}$. Since there are no a in the empty set, we conclude that $f(\varnothing) = \varnothing$.

By definition, $f^{-1}(\varnothing) = \{a \in A \mid f(a) \in \varnothing\}$. Since there are no $f(a)$ in the empty set, we conclude that $f^{-1}(\varnothing) = \varnothing$.

If $f^{-1}(B') = \varnothing$, then $f(a) \notin B'$ for all $a \in A$, *i.e.*, the function f completely misses the set B'. This is equivalent to $\operatorname{im} f \cap B' = \varnothing$. □

SOLUTION TO PROBLEM 5.6. For each equality $X = Y$, we need to demonstrate the two inclusions $X \subseteq Y$ and $Y \subseteq X$.

$f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$: If $y \in f(A_1 \cup A_2)$, then $y = f(x)$ for some $x \in A_1 \cup A_2$. If $x \in A_1$, then $y \in f(A_1)$, and if $x \in A_2$, then $y \in f(A_2)$. In either case, $y \in f(A_1) \cup f(A_2)$, proving that $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.

$f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$: Suppose $y \in f(A_1) \cup f(A_2)$. If $y \in f(A_1)$, then $y = f(x)$ for some $x \in A_1$; such an x is also in $A_1 \cup A_2$, so $y \in f(A_1 \cup A_2)$. If $y \in f(A_2)$, then $y = f(x)$ for some $x \in A_2$; such an x is also in $A_1 \cup A_2$, so $y \in f(A_1 \cup A_2)$. It follows that $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$: If $y \in f(A_1 \cap A_2)$, then $y = f(x)$ for some $x \in A_1 \cap A_2$. Such an x is in A_1 and A_2, and thus $y = f(x)$ is in $f(A_1)$ and $f(A_2)$, whence $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

$f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$: If $x \in f^{-1}(B_1 \cup B_2)$, then $f(x)\ in B_1 \cup B_2$, and thus $f(x) \in B_1$ or $f(x) \in B_2$. In the first case, $x \in f^{-1}(B_1)$; in the second case, $x \in f^{-1}(B_2)$. Thus $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$, and we conclude that $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$.

$f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$: If $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$, then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$. In the first case, $f(x) \in B_1 \subseteq B_1 \cup B_2$, so $f(x) \in B_1 \cup B_2$; similarly, in the second case $f(x) \in B_1 \cup B_2$. Thus always $x \in f^{-1}(B_1 \cup B_2)$ and we conclude that $f^{-1}(B_1) \cup f^{-1}(B_2) \subseteq f^{-1}(B_1 \cup B_2)$.

The final equality follows a similar line of argument. Make sure you can write out the proof on your own! □

5. Day 5

SOLUTION TO PROBLEM 6.1. (a) We compute

$$\begin{aligned} 2 \cdot (a_k a_{k-1} \dots a_2 a_1 a_0)_2 &= 2 \cdot (a_k 2^k + a_{k-1} 2^{k-1} + \dots + a_2 2^2 + a_1 2^1 + a_0 2^0) \\ &= a_k 2^{k+1} + a_{k-1} 2^k + \dots + a_2 2^3 + a_1 2^2 + a_0 2^1 \\ &= (a_k a_{k-1} \dots a_2 a_1 a_0 0). \end{aligned}$$

In other words, multiplying by 2 appends a 0 to the end of the binary representation. (Compare with multiplying by 10 in base 10.)

(b) Let B_n denote the binary number with n 1's. Thus $B_1 = 1_2 = 1$, $B_2 = 11_2 = 3$, $B_3 = 111_2 = 7$, *etc.*, and we conjecture that $B_n = 2^n - 1$. Indeed, if we add 1 to B_n (and use the usual addition algorithm with carrying) we get $B_n + 1 = 100 \dots 0_2$, where there are n 0's. Thus $B_n + 1 = 2^n$ and $B_n = 2^n - 1$.

□

SOLUTION TO PROBLEM 6.2 (FIRST METHOD). Label the elements of A as $a_1 = a$, a_2, …, a_n. Then we can encode subsets of A with n bit binary numbers where having first bit equal to 1 indicates $a \in A$. Thus we are seeking the number of n bit binary numbers with first bit equal to 1. We have two choices for each of the remaining $n - 1$ bits, and thus there are 2^{n-1} subsets of A containing a. □

SOLUTION TO PROBLEM 6.2 (SECOND METHOD). Let $X = \{B \subseteq A \mid a \in B\}$ and let Y be the set of subsets of $A \smallsetminus \{a\}$. Define a function $f : X \to Y$ by $f(B) = B \smallsetminus \{a\}$. (Note that $B \smallsetminus \{a\}$ is necessarily a subset of $A \smallsetminus \{a\}$, so the function is well-defined.) It suffices to prove now that f is a bijection.

To show injectivity, suppose $f(B) = f(C)$ for some $B, C \in X$. This means that $B \smallsetminus \{a\} = C \smallsetminus \{a\}$. Taking the union with $\{a\}$ on both sides gives $B = C$, so f is injective.

It remains to show that f is surjective. Given $C \subseteq A \smallsetminus \{a\}$, it is easy to check that $C \cup \{a\} \in X$ and $f(C \cup \{a\}) = C$, so f is surjective.

We conclude that f is a bijection, whence $|X| = |Y|$. Since Y is the set of subsets of a set of cardinality $n - 1$, both Y and X have cardinality 2^{n-1}. □

REMARK 5.1. The second solution method for Problem Problem 6.2 is an important one in combinatorics. Underlying it is the fact that two sets X and Y have the same cardinality if and only if there is a bijection $X \to Y$. If we know how to count the elements of Y and we can produce a bijection $X \to Y$, then we know X has the same number of elements!

SOLUTION TO PROBLEM 6.3. There are 3^n such pairs. Indeed, for each of the n elements of $\{1, \dots, n\}$, that element may be in neither A nor B, just in B, or in both A and B. Since there are three such choices for each element, there are 3^n pairs. □

SOLUTION TO PROBLEM 6.4. We can use ternary (*i.e.* base 3) numbers to easily enumerate the pairs. Ternary numbers consist of "trits" (trinary digits) taking the value 0, 1, or 2. We put a 0 for the k-th trit if k is in neither A nor B; a 1 for the k-th trit if k is in B but not in A; and a 2 for the k-th trit if k is in both B and A.

For $n = 3$, we get the dictionary

$$\begin{aligned} 000_3 &\longleftrightarrow \varnothing \subseteq \varnothing \subseteq \{1, 2, 3\} \\ 001_3 &\longleftrightarrow \varnothing \subseteq \{3\} \subseteq \{1, 2, 3\} \\ 002_3 &\longleftrightarrow \{3\} \subseteq \{3\} \subseteq \{1, 2, 3\} \\ 010_3 &\longleftrightarrow \varnothing \subseteq \{2\} \subseteq \{1, 2, 3\} \end{aligned}$$

$$
\begin{aligned}
011_3 &\longleftrightarrow \varnothing \subseteq \{2,3\} \subseteq \{1,2,3\}\\
012_3 &\longleftrightarrow \{3\} \subseteq \{2,3\} \subseteq \{1,2,3\}\\
020_3 &\longleftrightarrow \{2\} \subseteq \{2\} \subseteq \{1,2,3\}\\
021_3 &\longleftrightarrow \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\}\\
022_3 &\longleftrightarrow \{2,3\} \subseteq \{2,3\} \subseteq \{1,2,3\}\\
100_3 &\longleftrightarrow \varnothing \subseteq \{1\} \subseteq \{1,2,3\}\\
101_3 &\longleftrightarrow \varnothing \subseteq \{1,3\} \subseteq \{1,2,3\}\\
102_3 &\longleftrightarrow \{3\} \subseteq \{1,3\} \subseteq \{1,2,3\}\\
110_3 &\longleftrightarrow \varnothing \subseteq \{1,2\} \subseteq \{1,2,3\}\\
111_3 &\longleftrightarrow \varnothing \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
112_3 &\longleftrightarrow \{3\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
120_3 &\longleftrightarrow \{2\} \subseteq \{1,2\} \subseteq \{1,2,3\}\\
121_3 &\longleftrightarrow \{2\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
122_3 &\longleftrightarrow \{2,3\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
200_3 &\longleftrightarrow \{1\} \subseteq \{1\} \subseteq \{1,2,3\}\\
201_3 &\longleftrightarrow \{1\} \subseteq \{1,3\} \subseteq \{1,2,3\}\\
202_3 &\longleftrightarrow \{1,3\} \subseteq \{1,3\} \subseteq \{1,2,3\}\\
210_3 &\longleftrightarrow \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\}\\
211_3 &\longleftrightarrow \{1\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
212_3 &\longleftrightarrow \{1,3\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
220_3 &\longleftrightarrow \{1,2\} \subseteq \{1,2\} \subseteq \{1,2,3\}\\
221_3 &\longleftrightarrow \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}\\
222_3 &\longleftrightarrow \{1,2,3\} \subseteq \{1,2,3\} \subseteq \{1,2,3\}.
\end{aligned}
$$

□

SOLUTION TO PROBLEM 6.5. We can use the base $m+1$ number system to enumerate such chains. The $(m+1)$-ary digit ℓ in the k-th position indicates that k is in the sets $A_{m-\ell+1}, A_{m-\ell+2}, \ldots, A_m$, and that k is not in $A_1, \ldots, A_{m-\ell}$.

Since there are $m+1$ choices for each of the n $(m+1)$-ary digits, we see that there are $(m+1)^n$ such chains of subsets. □

6. Day 6

SOLUTION TO PROBLEM 7.1. There is a natural bijection explaining the co-incidence of the number k^n. Let X denote the set of length n strings with each entry coming from $\underline{k}$. For $s = s_1 s_2 \ldots s_n \in X$, let F_s denote the function $F_s : \underline{n} \to \underline{k}$ given by $F_s(a) = s_a$. Then the assignment $F : X \to \underline{k}^{\underline{n}}$ taking $s \mapsto F_s$ is a bijection, as we currently show.

Since X and $\underline{k}^{\underline{n}}$ have the same cardinality, it suffices to show that F is surjective. Given a function $f : \underline{k} \to \underline{n}$, define the string s by $s_a = f(a)$. Then $F_s(a) = s_a = f(a)$ for all $a \in \underline{n}$, so $F_s = f$, proving that F is surjective. □

REMARK 6.1. It can feel disorienting when you first work with functions between sets of functions. That's OK! Like an ouroboros, mathematics gains strength from devouring itself.

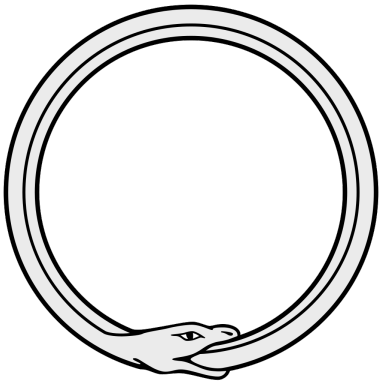


FIGURE 2. The ouroboros. (IMAGE: Wikipedia.)

SOLUTION TO PROBLEM 7.2. There are n^n strings of length n with entries in $\underline{n}$. Since permutations are special types of such strings (those with no repetition) and there are $n!$ permutations of $\underline{n}$, we conclude that $n! \leq n^n$.

We can rewrite $n!/n^n$ as

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n}.$$

Each factor is at most 1, and the last $n/2$ factors are smaller than $1/2$, so

$$\frac{n!}{n^n} \leq (1/2)^{n/2}$$

and the right-hand side goes to 0 as n goes to ∞.

We can interpret this as saying that there are vanishingly few permutations amongst all length n strings as n gets big. □

SOLUTION TO PROBLEM 7.3. For $n = 3$, the relevant pairs (i, j) are $(1, 2), (1, 3), (2, 3)$ and thus for a permutation $\pi : \underline{3} \to \underline{3}$, we have

$$\operatorname{sgn}(\pi) = \frac{\pi(2) - \pi(1)}{2 - 1} \cdot \frac{\pi(3) - \pi(1)}{3 - 1} \cdot \frac{\pi(3) - \pi(2)}{3 - 2}.$$

Let $S = \{(i, j) \mid i, j \in \mathbb{Z},\ 1 \leq i < j \leq n\}$ be the index set for the product, and let $\pi(S) = \{(\pi(i), \pi(j) \mid i, j \in \mathbb{Z},\ 1 \leq i < j \leq n\}$. The crucial observation is that for each $(i, j) \in S$, there is exactly one $(k, \ell) \in S$ such that either $(\pi(k), \pi(\ell))$ or $(\pi(\ell), \pi(k))$ is equal to (i, j). Reordering the numerators and denominators in the product expansion of $\operatorname{sgn}(\pi)$, we see that each $\frac{\pi(\ell)-\pi(k)}{j-i}$ is either 1 or -1, depending on whether the order of i and j was swapped by π. Thus the product as a whole is $(-1)^m$ where m is the number of pairs (i, j) with order swapped by π; in particular, $\operatorname{sgn}(\pi) = (-1)^m \in \{\pm 1\}$.

We now justify the crucial observation. For a given $(i, j) \in S$, we know there exist unique $k, \ell \in \underline{n}$ such that $\pi(k) = i$ and $\pi(\ell) = j$. If $k < \ell$, then $(k, \ell) \in S$ is the desired pair; if $k > \ell$, then $(\ell, k) \in S$ is the desired pair. □

7. Day 7

SOLUTION TO PROBLEM 9.1. The relation $\neq$ is not reflexive ($a \neq a$ is false), is symmetric (if $a \neq b$ then $b \neq a$), and is not transitive ($0 \neq 1$ and $1 \neq 0$, but $0 \neq 0$ is false).

The relation $>$ is not reflexive ($a > a$ is false), is not symmetric ($1 > 0$ but it is not the case that $0 > 1$), and is transitive.

The relation $\leq$ is reflexive, is not symmetric, and is transitive. □

SOLUTION TO PROBLEM 9.2. We check the properties one by one, beginning with reflexivity: if $x \in \mathbb{R}$, then $x - x = 0 \in \mathbb{Z}$, so $x \sim x$. For symmetry, suppose $x \sim y$, meaning that $x - y \in \mathbb{Z}$. Then $y - x = -(x - y)$ is an integer as well, so $y \sim x$. Finally, we check transitivty: if $x \sim y$ and $y \sim z$, then $x - y, y - z \in \mathbb{Z}$. Thus $(x - y) + (y - z) = x - z \in \mathbb{Z}$, so $x \sim z$. □

SOLUTION TO PROBLEM 9.3. Write $\sim$ for the relation defined in the problem. Without loss of generality, call the beads $1, 2, \ldots, n$, and write $a = a_1 a_2 \cdots a_n$ for a list of these beads. To say that $a \sim b$ is to say that for each $i \in \{1, \ldots, n\}$ there exists some $j \in \{1, \ldots, n\}$ such that $a_{i-1} = b_{j-1}$ and $a_{i+1} = b_{j+1}$ (where we interpret a_0 as a_n and interpret a_{n+1} as a_1), or $a_{i-1} = b_{j+1}$ and $a_{i+1} = b_{j-1}$.

LEMMA 7.1. For lists a, b, we have $a \sim b$ if and only if b is obtained from a either by rotating the indices of a cyclically, or by reversing the order of the indices and then rotating them cyclically.

(a) Reflexivity is obvious (right?). To check symmetry, suppose $a \sim b$. By the lemma, we can reverse the cycling/order-reversion that takes a to b to get $b \sim a$. To check transitivity, just note that composing two cycling/order-reversions gives a new cycling/order-reversion.
(b) There are n ways to cycle the indices of a given list (including the "do nothing" cycling). Each such cycling can be composed or not composed with order-reversion. Thus there are $2n$ lists in each equivalence class.
(c) Since each equivalence class has size $2n$ and there are $n!$ distinct lists, we have

$$\frac{n!}{2n}$$

total equivalence classes.

□

SOLUTION TO PROBLEM 9.4. Label the seats $1, \ldots, 2n$. Put Russians in seats $1, 3, \ldots, 2n - 1$ and put Americans in seats $2, 4, \ldots, 2n$. There are $n! \cdot n! = (n!)^2$ ways to do so. But we could have also put Russians in the even seats and Americans in the odd seats, so there are in fact $2(n!)^2$ total valid seatings. Declare two such seatings equivalent if one can be rotated to obtain the other. (We take it as obvious that this forms an equivalence relation, but it's good practice to check the conditions.) There are $2n$ such rotations, so there are

$$\frac{2(n!)^2}{2n} = (n-1)!n!$$

seating arrangements. □

ALTERNATE SOLUTION TO PROBLEM 9.4. Without loss of generality, assume that one of the Russians is named Natasha. We can choose a unique representative of each rotational equivalence class of seatings by selecting the seating with Natasha in seat 1. The remaining Russians must then go in seats $3, 5, \ldots, 2n - 1$, and the Americans can sit freely in seats $2, 4, \ldots, 2n$. This gives a direct count of $(n - 1)!n!$. □

SOLUTION TO PROBLEM 9.5. Again, it's fairly "obvious" that this is an equivalence relation. (But check!) In order to enumerate the equivalence classes, we will consider a word using RRBB to have first letter corresponding to the color in the northwest corner, second letter corresponing to

northeast corner, third corresponding to southeast, and fourth corresponding to southwest. Each word has up to four potentially distinct rotations:

$$RRBB \to RBBR \to BBRR \to BRRB$$
$$RBRB \to BRBR \to RBRB \to BRBR$$

We stop here because we've enumerated all the words in RRBB, but note that words are repeated in the second set of rotations. The equivalence classes are in fact

$$\{RRBB, RBRB, BBRR, BRRB\} \text{ and } \{RBRB, BRBR\}.$$

While it is the case that $2 = 6/3$, it is not the case that each equivalence class has size 3, so it would be inaccurate to say that we "found" the number of equivalence classes in this way. □

8. Day 8

SOLUTION TO PROBLEM 10.1. We first compute

$$\begin{aligned}
\binom{1}{0} &= 1 \\
\binom{2}{0} + \binom{2}{2} &= 2 \\
\binom{3}{0} + \binom{3}{2} &= 4 \\
\binom{4}{0} + \binom{4}{2} + \binom{4}{4} &= 8 \\
\binom{5}{0} + \binom{5}{2} + \binom{5}{4} &= 16 \\
\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} &= 32 \\
\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} &= 64.
\end{aligned}$$

Based on this evidence, we conjecture that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$. We can interpret this conjecture as saying that the number of even-sized subsets of a size n set is 2^{n-1}. Since the total number of subsets of such a set is 2^n, we could also say that precisely half of all subsets of a finite nonempty set have even size.

One nice argument for this fact relies on the decision tree model of creating a subset: Recall that the leaves of this binary tree correspond to the subsets. For each pair of leaves emanating from the final layer of nodes, exactly one has even and one has odd size. Thus half of all subsets have even size. □

SOLUTION TO PROBLEM 10.2. First we compute

$$\begin{aligned}
\binom{0}{0}^2 &= 1 \\
\binom{1}{0}^2 + \binom{1}{1}^2 &= 2 \\
\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 &= 6 \\
\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 &= 20 \\
\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 &= 70 \\
\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2 &= 252
\end{aligned}$$

Suspciously and amazingly, these appear in the center column of Pascal's triangle as the numbers of the form $\binom{2n}{n}$. We conjecture that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

and note that this matches the cases computed above.

This puts us on the hunt for subsets of a size $2n$ set of size n. Suppose $|A| = 2n$ and then color half its elements blue and half its elements red. (We can do that!) To get a size n subset of A, we can choose a blue elements and b red elements where $a + b = n$. For fixed a, there are $\binom{n}{a}\binom{n}{b}$ ways to do this. Since $b = n - a$, we have $\binom{n}{b} = \binom{n}{n-a} = \binom{n}{a}$, and so $\binom{n}{a}\binom{n}{b} = \binom{n}{a}^2$. Letting a vary from 0 to n, we see that in sum we have

$$\begin{aligned}
\binom{2n}{n} &= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} \\
&= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2,
\end{aligned}$$

as desired. □

9. Day 9

SOLUTION TO PROBLEM 11.1. After playing around for a while (OK, maybe a long while...), one comes to the conclusion that $\binom{m-1}{r-1}$ gives the desired count. For instance, we can represent 5 as the sum of 3 positive integers as $3+1+1$, $1+3+1$, $1+1+3$, $2+2+1$, $2+1+2$, or $1+2+2$, and $6 = \binom{4}{2}$.

A nice argument for this is given by the Balls and Walls method.[1] Imagine that we have m balls in a row. In order to represent m as a sum of r positive integers, we can place $r-1$ walls in the spaces between the balls, taking care to not place two or more walls in a single gap. For example, the sum $7 = 1+3+2+1$ is represented by

$$\bullet | \bullet\bullet\bullet | \bullet\bullet | \bullet .$$

[1] *Née* Stars and Bars, but that's a little too militaristic for Reed in my opinion.

There is clearly a bijection between such ball-wall configurations and the sums we are counting, and each ball-wall configuration is specified by choosing $r-1$ spots to places walls amongst the $m-1$ gaps between balls; this number is, of course, $\binom{m-1}{r-1}$. □

SOLUTION TO PROBLEM 11.2. Let x be a variable. By the binomial theorem

$$(1+x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i.$$

In particular, the coefficient of x^n in this polynomial is $\binom{2n}{n}$.

We also have $(1+x)^{2n} = (1+x)^n(x+1)^n$, and applying the binomial theorem to each factor results in

$$(1+x)^{2n} = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^{n-j}\right).$$

When we expand this product, we get a term contributing to x^n when $i+n-j=n$, *i.e.* when $i=j$. Thus the coefficient of x^n is $\sum_{i=0}^{n} \binom{n}{i}^2$, and this must equal our alternate computation of the coefficient, $\binom{2n}{n}$. □

COMBINATORIAL SOLUTION TO PROBLEM 11.3. The summands on the left-hand side are suggestive of first choosing k elements from a size n set, and then choosing m elements from the k elements. This could be modeled by choosing a size k committee from n members, and then choosing a size m subcommittee of the committee. Since we are summing these over $k = 0, 1, \ldots, n$ while m is fixed, this counts the number of committees formed from $\{1, \ldots, n\}$ with a size m subcommittee. We can also count this by first choosing the size m subcommittee (in $\binom{n}{m}$ possible ways) and then choosing a subset of the remaining elements to form the remainder of the committee. Since there are $n-m$ remaining members, there are 2^{n-m} such subsets, and we conclude that there are $\binom{n}{m}2^{n-m}$ committe-with-size-m-subcommittee pairs from n members. Since both sides count the same thing, they are equal. □

ALGEBRAIC SOLUTION TO PROBLEM 11.3. As the hint suggests, first note that

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{k!(n-k)!} \cdot \frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!}$$

while

$$\binom{n}{m}\binom{n-m}{k-m} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \frac{n!}{m!(k-m)!(n-k)!}.$$

These quantities are obviously equal, so $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.[2]

We now leverage the identity $2^{n-m} = \sum_{i=0}^{n-m} \binom{n-m}{i}$ to manipulate the right-hand side, first noting that $\sum_{i=0}^{n-m} \binom{n-m}{i} = \sum_{k=m}^{n} \binom{n-m}{k-m}$ via the change of variables $i = k-m$. (Check that the summands are in fact identical.) Thus we have

$$\binom{n}{m}2^{n-m} = \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{k-m} = \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}$$

where the second equality uses the hint's identity. When $k < m$, $\binom{k}{m} = 0$, so the final sum can also be indexed with k ranging from 0 to n, producing the desired identity. □

[2] It is also possible to give a combinatorial argument for this equality: The left-hand side counts k-subsets of an n-set paired with an m-subset of the k-subset. The right-hand side counts the m-subset first and then chooses the remaining $k-m$ members of the k-subset from the remaining $n-m$ elements of the n-set.

10. Day 10

SOLUTION TO PROBLEM 12.1. By the n-th diagonal, we mean $\binom{n}{0}, \binom{n+1}{1}, \binom{n+2}{2}, \binom{n+3}{3}, \ldots$. By iterated application of Pascal's identity, we know that $\binom{n+k}{k}$ is the sum of the preceding elements on the $(k-1)$-th diagonal, *i.e.*,

$$\binom{n+k}{k} = \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k}.$$

In particular, the second diagonal consists of the sums of consecutive positive integers,

$$\binom{2+k}{k} = \binom{1}{0} + \binom{2}{1} + \binom{3}{2} + \cdots + \binom{1+k}{k} = 1 + 2 + 3 + \cdots + (k+1).$$

These numbers are sometimes called the *triangular numbers*. (Note that $\binom{2+k}{k} = \binom{2+k}{2}$, so we can also write $\binom{n}{2} = 1 + 2 + \cdots + (n-1)$.) □

SOLUTION TO PROBLEM 12.2. We have

$$\begin{aligned}\binom{n}{2} &= 1 + 2 + 3 + \cdots + (n-3) + (n-2) + (n-1) \\ \binom{n+1}{2} &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.\end{aligned}$$

Since $0 + n = 1 + (n-1) = 2 + (n-2) = 3 + (n-3) = \cdots = n$ (note the vertical alignment above), and there are n such terms, we have that $\binom{n}{2} + \binom{n+1}{2} = n \cdot n = n^2$. □

SOLUTION TO PROBLEM 12.3. Let $\alpha(n)$ denote the number of 1's in the binary expansion of n. Let $O(n)$ denote the number of odd numbers in the n-th row of Pascal's triangle. We claim that $O(n) = 2^{\alpha(n)}$.[3]

To present a good proof of this fact, we'll need modular arithmetic, specifically mod 2 arithmetic. We defer the proof until we've developed that technology. □

11. Day 11

SOLUTION TO PROBLEM 13.1. For $n = 1$, we have $2^0 = 1 = 2^1 - 1$, so the base case checks. Now fix some $n \geq 1$ and suppose that $2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1$. (This is our inductive hypothesis.) Then

$$2^0 + 2^1 + \cdots + 2^{n-1} + 2^n = 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$$

so the result holds for $n+1$ as well. By mathematical induction, the identity holds for all $n \geq 1$. □

SOLUTION TO PROBLEM 13.2. If $n = 0$, then $\underline{n} = \underline{0} = \varnothing$ and there is only one permutation of $\varnothing$. Since $0! = 1$, this confirms the base case $n = 0$. Now fix $n \geq 0$ and suppose for induction that there are $n!$ permutations of $\underline{n}$. Now think of a permutation π of $\underline{n+1}$ as its list of outputs, $\pi(1)\pi(2)\cdots\pi(n)\pi(n+1)$. All such lists arise by first permuting $\underline{n}$ (in any of the $n!$ ways) and then placing $n+1$ at the start of the list, in between two numbers, or at the end of the list. There are $n+1$ such positions and hence $n!(n+1) = (n+1)!$ permutations of $\underline{n+1}$. □

[3] How would you ever guess such a result?! Patience and experimentation, for starters. You might first get some hunches by seeing that (a) the first several values of $O(n)$ are $1, 2, 2, 4, 2, 4, 4, 8, 2, 4, \ldots$, and these are all powers of 2, (b) $O(2^k)$ seems to always be $2 = 2^1$, and (c) $O(2^k - 1)$ seems to always be 2^k.

SOLUTION TO PROBLEM 13.3. If $n = 1$, then $\frac{1}{1\cdot 2} = \frac{1}{2} = \frac{1}{1+1}$, so the base case holds. Now fix $n \geq 1$ and suppose for induction that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. Then

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \\ &= \frac{n^2+2n+1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2},\end{aligned}$$

as desired. □

SOLUTION TO PROBLEM 13.4. Our base case is $n = 3$, the triangle, which has no diagonals, and indeed $3(3-3)/2 = 0$. Fix $n \geq 3$ and suppose for induction that that a convex n gon has $n(n-3)/2$ diagonals. Now consider a convex $(n+1)$-gon with vertices labeled $1, 2, \ldots, n+1$ in order. By the inductive hypothesis, the n-gon with vertices $1, \ldots, n$ has $n(n-3)/2$ diagonals, and each of these is a diagonal of our $(n+1)$-gon. Additionally, the $(n+1)$-gon has diagonals joining $n+1$ to $2, 3, \ldots, n-1$, and it also has the diagonal from 1 to n. That amounts to $n-1$ additional diagonals, so the $(n+1)$-gon has

$$\frac{n(n-3)}{2} + n - 1 = \frac{(n^2-3n)+(2n-2)}{2} = \frac{(n+1)((n+1)-3)}{2}$$

diagonals, as desired. □

SOLUTION TO PROBLEM 13.5. When $n = 5$, we have $\binom{10}{5} = 252$ and $2^8 = 256$, so the inequality holds in the base case. Fix $n \geq 5$ and assume for induction that $\binom{2n}{n} < 2^{2n-2}$. Since $2^{2(n+1)-2} = 2^{2n} = 4 \cdot 2^{2n-2}$, it suffices to prove that $\binom{2(n+1)}{n+1} < 4\binom{2n}{n}$. By algebra,

$$\binom{2(n+1)}{n+1} = \frac{(2n+2)!}{(n+1)!^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{(2n)!}{n!^2} = \frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n},$$

so it suffices to prove that $\frac{(2n+2)(2n+1)}{(n+1)^2} < 4$. This is the case if and only if $(2n+2)(2n+1) < 4(n+1)^2$, *i.e.*, $4n^2 + 6n + 2 < 4n^2 + 8n + 4$, *i.e.*, $0 < 2n + 2$, which is in fact true for all natural numbers n. □

REMARK 11.1. The reason the theorem does not extend to all natural numbers is because the base case does not hold until $n = 5$.

12. Day 12

SOLUTION TO PROBLEM 14.2. Let S, F, and R denote the sets of Spanish, French, and Russian students, respectively. We are given that

$$|S| = 1232, \qquad |F| = 879, \qquad |R| = 114,$$

and

$$|S \cap F| = 103, \qquad |S \cap R| = 23, \qquad |F \cap R| = 14.$$

Furthermore, $|S \cup F \cup R| = 2092$. By the inclusion-exclusion principle,

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

so

$$\begin{aligned}|S \cap F \cap R| &= |S \cup F \cup R| - |S| - |F| - |R| + |S \cap F| + |S \cap R| + |F \cap R| \\ &= 2092 - 1232 - 879 - 114 + 103 + 23 + 14 \\ &= 7,\end{aligned}$$

and this is the number of students taking a course in all three languages. □

SOLUTION TO PROBLEM 14.3. Let S denote the set of hands with at least one card from each suit, and let H denote the set of all hands. Then $S = H \setminus (N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit)$ and $|S| = |H| - |N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|$. Since each hand contains 5 of the 52 cards, $|H| = \binom{52}{5}$, and it remains to count $|N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit|$.

We proceed via inclusion-exclusion. Since only the excluded suit changes, we have $|N_\spadesuit| = |N_\clubsuit| = |N_\heartsuit| = |N_\diamondsuit|$, and for each of these counts we select 5 cards from the $52 - 13 = 39$ cards which aren't of the selected suit. Thus the cardinality of each of these is $\binom{39}{5}$. Each pairwise intersection excludes 26 cards and thus has cardinality $\binom{26}{5}$, and each triple intersection excludes 39 cards and thus has cardinality $\binom{13}{5}$. The quadruple intersection is empty, since each card has some suit. Note that there are $\binom{4}{2} = 6$ pairwise intersections and there are $\binom{4}{3} = 4$ triple intersections. We conclude that

$$|N_\spadesuit \cup N_\clubsuit \cup N_\heartsuit \cup N_\diamondsuit| = 4 \cdot \binom{39}{5} - 6 \cdot \binom{26}{5} + 4 \cdot \binom{13}{5}$$

and

$$|S| = \binom{52}{5} - 4 \cdot \binom{39}{5} + 6 \cdot \binom{26}{5} - 4 \cdot \binom{13}{5} = 685,464.$$

□

ALTERNATE SOLUTION TO PROBLEM 14.3. We can also proceed without using the inclusion-exclusion principle. Every such hand can be constructed by choosing a spade, then a club, then a heart, then a diamond, and then one of the remaining 48 cards. This results in $13^4 \cdot 48$ choices, but overcounts in that the final card may be swapped with the other card of its suit, resulting in the same hand. (Hands don't have an order.) Thus there are

$$\frac{13^4 \cdot 48}{2} = 685,464$$

such hands. □

SECOND ALTERNATE SOLUTION TO PROBLEM 14.3. In order to construct such a hand, we first choose any of the 52 cards and note its suit. We then choose any of the remaining 39 cards of a different suit, then any of the remaining 26 cards not of the first two suits, then any of the remaining 13 cards not of the first 3 suits. Finally, we choose any of the remaining 48 cards. All such hands can be produced in this way, but there are still $4!$ to permute the first four cards and 2 ways to swap (or not swap) the final card with the one matching its suit. Thus there are

$$\frac{52 \cdot 39 \cdot 26 \cdot 13 \cdot 48}{4! \cdot 2} = 685,464$$

such hands. □

13. Day 13

SOLUTION TO PROBLEM 15.1. At any given moment, each player has played between 0 and $n-1$ games, a range of n possibilities, so the pigeonhole principle does not directly apply. Note, though, that if one player has played $n-1$ games, then everyone has played between 1 and $n-1$ games, a range of $n-1$ possibilities. If no players have played $n-1$ games, then everyone has played between 0 and $n-2$ games, again $n-1$ possibilities. Thus the pigeonhole principle applies in both cases to guarantee that (at least) two players have played the same number of games. □

SOLUTION TO PROBLEM 15.2. There are $8 \cdot 10^6$ seven-digit phone numbers (excluding area code) according to these rules. With 3 or fewer area codes, there are at most 24 million distinct phone numbers, whence the pigeonhole principle would guarantee phone number repetition in the state. With 4 area codes, there are 32 million distinct phone numbers, a sufficient number to prevent repetition. □

SOLUTION TO PROBLEM 15.3. Following the hint, suppose $a_i > a_j$ are terms of the sequence such that $a_i - a_j$ is divisible by 2003. The number $a_i - a_j$ is of the form $a_k \cdot 10^r$ for some positive integer r. Since 2003 does not share any prime factors with 10 (in fact, 2003 is prime), we have that 2003 divides a_k.

Now note that when we divide a term a_i by 2003, we get a remainder between 0 and 2002. If the remainders of terms a_i and a_j are equal, then $a_i - a_j = 2003q_i + r - (2003q_j + r) = 2003(q_i - q_j)$ for some integers q_i, q_j, r. Thus 2003 divides $a_i - a_j$. Finally, note that there are finitely many remainders and infinitely many terms $a_i > a_j$, so such a pair with common remainder must exist. □

14. Day 14

SOLUTION TO PROBLEM 17.1. If $\pi(1) = 2$ and $\pi(2) = 1$, then the restriction of π to $\{3, 4, \ldots, n\}$ is a derangment of an $(n-2)$-element set, and all such derangments arise in this way. Thus there are $(n-2)_¡$ derangments of this form. The same argument applies to derangements with $\pi(1) = k$ and $\pi(k) = 1$, with $\{i \mid i \in \mathbb{N},\ 2 \le i \le n,\ i \ne k\}$ playing the role of $\{3, 4, \ldots, n\}$. □

SOLUTION TO PROBLEM 17.2. If $\pi(1) = 2$, and $\pi(2) \ne 1$, then "the rest" of π (meaning the restriction of π to $\{2, 3, \ldots, n\}$) constitutes a bijection $\pi' : \{2, 3, 4, \ldots, n\} \to \{1, 3, 4, \ldots, n\}$. This bijection satisfies $\pi'(2) \ne 1$, $\pi'(3) \ne 3$, $\pi'(4) \ne 4$, …, $\pi'(n) \ne n$, *i.e.*, each element of the domain has one excluded outcome. This is the same as counting the number of derangements of an $(n-1)$-element set, $(n-1)_¡$. The same argument applies to any other fixed k, $2 \le k \le n$ and π such that $\pi(1) = k$, $\pi(k) \ne 1$. □

SOLUTION TO PROBLEM 17.3. Given a derangment π of $\underline{n}$, we have $\pi(1)$ equal to some k, $2 \le k \le n$, and there are $n-1$ such k. Either $\pi(k) = 1$, and there are $(n-2)_¡$ such derangments for each k, or $\pi(k) \ne 1$, and there are $(n-1)_¡$ such derangements for each k. We conclude that $n_¡ = (n-1) \cdot (n-2)_¡ + (n-1) \cdot (n-1)_¡$, or, more compactly,

$$n_¡ = (n-1)((n-2)_¡ + (n-1)_¡).$$

By direct inspection, we have $1_¡ = 0$ and $2_¡ = 1$. Thus

$$\begin{aligned} 3_¡ &= 2(0+1) = 2, \\ 4_¡ &= 3(1+2) = 9, \\ 5_¡ &= 4(2+9) = 44, \\ 6_¡ &= 5(9+44) = 265. \end{aligned}$$

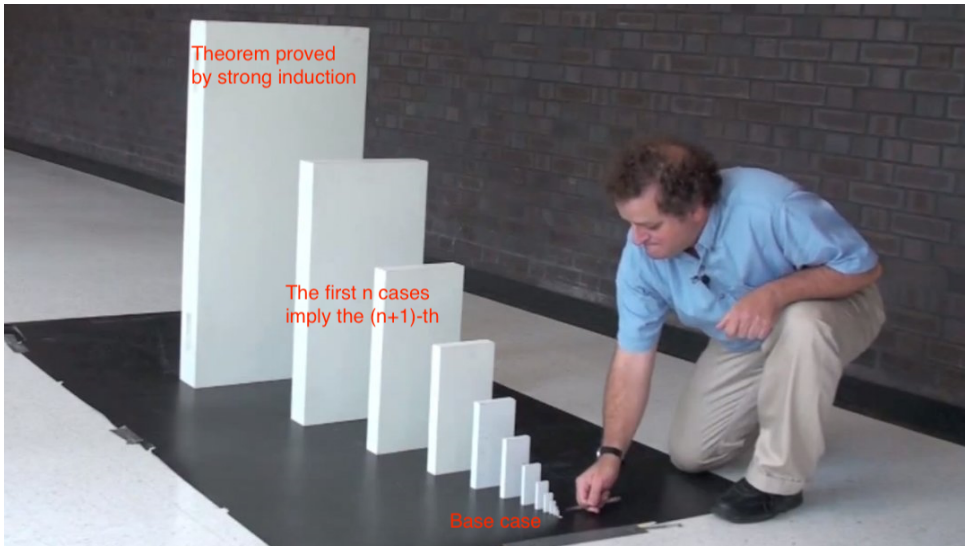


FIGURE 3. A set of dominoes for which the strong induction hypothesis is necessary?

We also have

$$3!(1 - 1/1! + 1/2! - 1/3!) = 3 - 1 = 2,$$
$$4!(1 - 1/1! + 1/2! - 1/3! + 1/4!) = 12 - 4 + 1 = 9,$$
$$5!(1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5!) = 60 - 20 + 5 - 1 = 44,$$
$$6!(1 - 1/1! + 1/2! - 1/3! + 1/4! - 1/5! + 1/6!) = 360 - 120 + 30 - 6 + 1 = 265,$$

as expected. □

15. Day 15

SOLUTION TO PROBLEM 18.1. Let D_n be the number of ways to fill a $2 \times n$ chessboard with 2×1 dominoes. By inspection, we see that $D_1 = 1$, $D_2 = 2$, $D_3 = 3$, $D_4 = 5$, and $D_5 = 8$. We thus suspect that $D_n = F_{n+1}$ for $n \geq 1$.

We proceed by strong induction,[4] having already verified the first several base cases. Now fix $n \geq 2$ and suppose that $D_n = F_{n+1}$ ad $D_{n-1} = F_n$. In a $2 \times (n+1)$ chessboard, the top right square must be covered by a horizontal or a vertical domino. In the first case, another horizontal domino must be directly below the top right one, and thus it remains to fill a $2 \times (n-1)$ board with $n-1$ dominoes. By the strong induction hypothesis, we can do this in $D_{n-1} = F_n$ many ways. In the vertical case, it remains to fill a $2 \times n$ board with n dominoes, which we can do in $D_n = F_{n+1}$ many ways. Since the cases are mutually exclusive, we conclude that the board may be filled in

$$D_{n+1} = F_n + F_{n+1} = F_{n+2}$$

many ways, finishing our proof. □

SOLUTION TO PROBLEM 18.2. Let S_n denote the sum in question when we begin with $\binom{n}{0}$. Then $S_0 = 1$, $S_1 = 1$, $S_2 = 2$, $S_3 = 3$, $S_4 = 5$, $S_5 = 8$, and $S_6 = 13$. We suspect that $S_n = F_{n+1}$.

[4] In strong induction, your induction hypothesis is that for some n, the claim holds for that n and all previous n; you then show that this hypothesis implies the claim for $n + 1$. See Figure 3.

To prove this, we need to check that $S_0 = F_1$, $S_1 = F_2$, and $S_{n-1} + S_n = S_{n+1}$ for $n \geq 1$. We have already seen the first two facts.

Fix $n \geq 1$. By definition, $S_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{0}{n} = \sum_{k=0}^{n} \binom{n-k}{k}$. (We have extended the sum into the "0-range" of Pascal's triangle in order to make the indexing easier.) Then

$$S_{n-1} + S_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n} \binom{n-k}{k}$$

In the first sum, we can allow the indices to range from 0 to n by replacing k with $k-1$. (The first term becomes $\binom{n}{-1} = 0$, which is fine. Also note that the upper term of the binomial coefficient becomes $n - 1 - (k-1) = n - k$.) Thus

$$\begin{aligned} S_{n-1} + S_n &= \sum_{k=0}^{n} \binom{n-k}{k-1} + \sum_{k=0}^{n} \binom{n-k}{k} \\ &= \sum_{k=0}^{n} \binom{n-k}{k-1} + \binom{n-k}{k} \\ &= \sum_{k=0}^{n} \binom{(n+1)-k}{k} \end{aligned}$$

where the third equality follows from Pascal's identity. This final quantity is missing the $\binom{0}{n+1}$ term from our definition of S_{n+1}, but this is 0 so the two quantities are equal. We have shown that $S_{n-1} + S_n = S_{n+1}$, so our proof is complete. □

SOLUTION TO PROBLEM 18.3. We have $F_{-1} = F_1 - F_0 = 1$, $F_{-2} = F_0 - F_{-1} = -1$, $F_{-3} = F_{-1} - F_{-2} = 2$, $F_{-4} = F_{-2} - F_{-3} = -3$, and $F_{-5} = F_{-3} - F_{-4} = 5$. It appears that $F_{-n} = (-1)^{n-1} F_n$ for $n \geq 1$. The base case has been checked and, for a proof by strong induction, we fix $n \geq 2$ and assume $F_{-n} = (-1)^{n-1} F_n$ and $F_{-(n-1)} = (-1)^{n-2} F_{n-1}$. By definition, $F_{-(n+1)} = F_{-(n-1)} - F_{-n} = (-1)^{n-2} F_{n-1} - (-1)^{n-1} F_n = (-1)^{n-2}(F_{n-1} + F_n) = (-1)^n F_{n+1}$, where the last equality uses the recursive definition of the Fibonacci sequence and the fact that $(-1)^n = (-1)^{n-2}$ for all n. This concludes our proof by strong induction. □

16. Day 16

SOLUTION TO PROBLEM 19.1. We have $G_1 = 1$, $G_2 = 3$, $G_3 = 8$, $G_4 = 21$, $G_5 = 55$. These are all Fibonacci numbers, and after fiddling with indices for long enough, it appears that $G_n = F_{2n}$. We have checked the first several cases and, for a proof by induction, we fix $n \geq 1$ and assume that $G_n = F_{2n}$. Then $G_{n+1} = G_n + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2} = F_{2(n+1)}$, concluding our proof. □

SOLUTION TO PROBLEM 19.2. For $n \geq 1$, let $H_n = F_{n-1} F_{n+1} - F_n^2$. Then

$$\begin{aligned} H_1 &= 0 \cdot 1 - 1^2 = -1 \\ H_2 &= 1 \cdot 2 - 1^2 = 1 \\ H_3 &= 1 \cdot 3 - 2^2 = -1 \\ H_4 &= 2 \cdot 5 - 3^2 = 1 \\ H_5 &= 3 \cdot 8 - 5^2 = -1. \end{aligned}$$

It appears that $H_n = (-1)^n$. We have verified $H_1 = -1$. For induction, fix $n \geq 1$ and assume $H_n = (-1)^n$. Then

$$\begin{aligned} H_{n+1} &= F_{n+2}F_n - F_{n+1}^2 \\ &= (F_n + F_{n+1})F_n - F_{n+1}^2 \\ &= F_n^2 + (F_n - F_{n+1})F_{n+1} \\ &= F_n^2 - F_{n-1}F_{n+1} \\ &= -H_n = -(-1)^n = (-1)^{n+1}, \end{aligned}$$

as desired. □

17. Day 17

SOLUTION TO PROBLEM 20.1. (a) We see that $a_0 = 1$, $a_1 = 4$, and $a_3 = 8$.

(b) It is tempting to conjecture that $a_n = 2^n$, but from our picture we see that $a_4 = 14$.

(c) The new circle intersects each of the $n-1$ circles in two points, so there are a total of $2(n-1)$ intersections. This produces $2(n-1)$ arcs on the new circle.

(d) Each arc splits an old region into two regions, *i.e.*, creates one new region. Thus a_n satisfies the recurrence $a_n = a_{n-1} + 2(n-1)$ for $n \geq 2$. (Our analysis in (c) depended on there being at least one circle in the $(n-1)$-th case.) Thus a_n is given by the initial conditions $a_0 = 1$, $a_1 = 2$, and the above recurrence.

(e) Iteratively applying the recurrence relation to a_n when $n \geq 2$ results in

$$\begin{aligned} a_n &= a_{n-1} + 2(n-1) \\ &= a_{n-2} + 2(n-2) + 2(n-1) \\ &= a_{n-3} + 2(n-3) + 2(n-2) + 2(n-1) \\ &\vdots \\ &= a_1 + 2(1) + 2(2) + 2(3) + \cdots + 2(n-2) + 2(n-1) \\ &= 2 + 2(1 + 2 + \cdots + (n-1)) \\ &= 2 + n(n-1) \\ &= n^2 - n + 2. \end{aligned}$$

Here we employed the identity $1 + 2 + \cdots + (n-1) = n(n-1)/2$ to get the second-to-last equality. This proves that $a_n = n^2 - n + 2$ for $n \geq 2$. By coincidence, the identity holds for $n = 1$ as well, but does not hold for $n = 0$.

□

SOLUTION TO PROBLEM 20.2. (a) Each path ends in some cell, and by symmetry the same number of paths, a_n end in cells 1, 3, 7, and 9; similarly, the same number of paths, b_n, end in 2, 4, 6, and 8; the remaining case is the c_n paths ending in cell 5. Thus $p_n = 4a_n + 4b_n + c_n$.

(b) In order to end in cell 1 in n steps, the ant may either be in cell 2 or 4 at step $n-1$. Thus $a_n = 2b_{n-1}$. To end in cell 2 in n steps, the ant may either be in cell 1, 3, or 5 at step $n-1$. Thus $b_n = 2a_{n-1} + c_{n-1}$. Finally, $c_n = 4b_{n-1}$ since to end in cell 5 in n steps, the ant must be in cell 2, 4, 6, or 8 at step $n-1$. Our system of recurrences is

$$\begin{aligned} a_n &= 2b_{n-1} \\ b_n &= 2a_{n-1} + c_{n-1} \\ c_n &= 4b_{n-1}. \end{aligned}$$

(c) Since $a_{n-1} = 2b_{n-2}$ and $c_{n-1} = 4b_{n-2}$, we get

$$b_n = 2 \cdot 2b_{n-2} + 4b_{n-2} = 8b_{n-2}.$$

Since $b_1 = 3$ and $b_2 = 8$, we can solve for b_n explicitly as $b_n = 8^{n/2} \cdot 8 = 8^{n/2+1}$ if n is even and $b_n = 8^{(n-1)/2} \cdot 3$ if n is odd.

(d) We know that $p_n = 4a_n + 4b_n + c_n$ for $n \geq 1$, which becomes $p_n = 4 \cdot 2b_{n-1} + 8b_{n-2} + 4b_{n-1} = 12b_{n-1} + 8b_{n-2}$ for $n \geq 2$. Thus if n is even and ≥ 2, $p_n = 12 \cdot 8^{(n-2)/2} \cdot 3 + 8 \cdot 8^{(n-2)/2+1} = 36 \cdot 8^{(n/2-1} + 8^{n/2+1}$. If n is odd and ≥ 2, $p_n = 12 \cdot 8^{(n-1)/2+1} + 8 \cdot 8^{(n-3)/2} \cdot 3 = 12 \cdot 8^{(n-1)/2+1} + 3 \cdot 8^{(n-3)/2+1}$.

(e) The explicit computations are not horribly illuminating, but the asymptotic growth is proportional to $8^{n/2}$, which is exponential. In reality, the ant is paralyzed by the overwhelming number of choices and simply stays put.

□

18. Day 18

SOLUTION TO PROBLEM 21.1. There are $\binom{n}{2}$ edges in K_n, since there are as many edges as there are choices of 2 vertices. Since $K_{|V|}$ has the maximal number of edges amongst graphs with $|V|$ vertices, we know that $|E| \leq \binom{|V|}{2}$ for a general graph $G = (V, E)$. □

SOLUTION TO PROBLEM 21.2. (a)

(b) Each of the p vertices in A is connected to all q vertices in B, so $K_{p,q}$ has pq edges.

(c) Each of the pq potential edges joining A to B is either in or not in the graph. Thus there are 2^{pq} such bipartite graphs.

□

SOLUTION TO PROBLEM 21.3. (a) The function must take edges to edges, so we require that if $\{v, w\} \in E$, then $\{f(v), f(w)\} \in E'$.

(b) We demand that there exist maps of graphs $f : G \to G'$ and $g : G' \to G$ such that $f \circ g = \mathrm{id}_{V'}$ and $g \circ f = \mathrm{id}_V$. Thus f is a bijection on the set of vertices, it preserves edges, and its inverse function also preserves edges. This is equivalent to f being a bijection on vertex sets which induces a bijection on edge sets $\{v, w\} \mapsto \{f(v), f(w)\}$.

□

19. Day 19

SOLUTION TO PROBLEM 22.1. We must show that there is a walk in G between any two vertices in G. Given $v, w \in V$, such a walk exists if both vertices are in V_1 or both are in V_2 since H_1 and H_2 are connected. Now suppose that $v \in H_1$ and $w \in H_2$. Choose $u \in V_1 \cap V_2$. By connectivity of H_1, there is a walk in G from v to u. By connectivity of H_2, there is a walk in G from u to w. Concatenating those paths, we get a walk from v to w, as desired. □

SOLUTION TO PROBLEM 22.2. We prove that there are $n - 1$ edges in a tree with n vertices by induction on $n \geq 1$. Clearly, if $n = 1$ then there are $0 = 1 - 1$ edges in a single vertex tree. For induction, fix $n \geq 1$ and suppose that every tree with n vertices has $n - 1$ edges. Given a tree with $n + 1$ vertices, there exists a vertex of degree 1 (why?). Prune this vertex and its edge from the tree to get a tree with n vertices and hence $n - 1$ edges. The $(n + 1)$-vertex tree has one more edge, hence $n = (n + 1) - 1$ edges, as desired. □

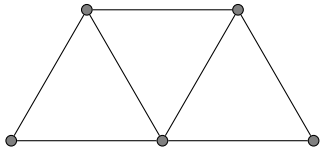
SOLUTION TO PROBLEM 22.3. First suppose that G is a tree. Since G is connected, there is at least one path between any two vertices. Suppose for contradiction that there are two paths $P_1 \neq P_2$ joining $u \neq v \in G$. Suppose P_1 goes from $u = u_1$ to u_2 to u_3 to $\ldots$ to $u_k = v$ and P_2 goes

from $u = v_1$ to v_2 to v_3 to ... to $v_\ell = v$. Let i be the first index such that $u_i \neq v_i$ and let $j \geq i$ be the next index so that $u_j = v_m$ for some $i \leq m \leq \ell$. Then we have paths from u_{i-1} to u_j and (reversing part of P_2) from $u_j = v_m$ to $u_{i-1} = v_{i-1}$. This creates a circuit, contradicting the hypothesis that G is a tree.

Now suppose that G is not a tree. Then either G is not connected (in which case there are vertices joined by no path) or G contains a cycle $u_0u_1u_2 \cdots u_mu_0$. Then u_0u_1 and $u_0u_mu_{m-1} \cdots u_2u_1$ are two distinct paths from u_0 to u_1. □

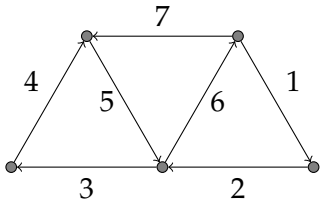
20. Day 20

SOLUTION TO PROBLEM 23.1. (a) We may think of each room as a vertex, and then connect rooms with edges if they share an interior wall. This results in the graph



for which we would like to know if there is an Eulerian walk.

(b) The vertices have degrees 2, 4, 2, 3, and 3 starting from the lower left and moving counter-clockwise. Thus an Eulerian walk exists and it must start at one of the upper two vertices and end at the other. Here is an example of such a walk:



(c) Considering the exterior walls introduces an extra vertex in the graph corresponding to the exterior of the building joined to the bottom vertex by a single edge and joined to every other edge by **two** edges. Thus the new vertex has degree 9 and the top two vertices each have degree 5. We conclude that this graph has no Eulerian walks since more than two vertices have odd degree.

□

21. Day 21

SOLUTION TO PROBLEM 24.1. The only full binary tree with 1 leaf is the singleton tree, of which there is 1, so $C_0 = 1$.

Given a full binary tree T with $n+1$ leaves, $n \geq 0$, let $L(T)$ denote its left sub-tree (with root the left child of the root of T and all its children in T) and let $R(T)$ denote its right sub-tree (similarly defined). Then $L(T)$ has $1 \leq j \leq n+1$ leaves and $R(T)$ has $n+2-j$ leaves. The number of possibilities for $L(T)$ with j leaves is counted by C_{j-1}, and then there are C_{n+1-j} possibilities for $R(T)$. This proves that

$$C_{n+1} = \sum_{j=1}^{n+1} C_{j-1}C_{n+1-j}.$$

Changing indices with $i = j - 1$ gives

$$C_{n+1} = \sum_{i=0}^{n} C_iC_{n-i}.$$

By the recurrence,

$$\begin{aligned}
C_1 &= C_0C_0 = 1,\\
C_2 &= C_0C_1 + C_1C_0 = 2,\\
C_3 &= C_0C_2 + C_1C_1 + C_2C_0 = 5,\\
C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 14,\\
C_5 &= C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 42.
\end{aligned}$$

□

SOLUTION TO PROBLEM 24.2. Call the factors $a_1, \ldots, a_{n+1}$ and label the leaves with the factors from left to right. Call the *level* of a node k if it is k steps from the root. Begin with the largest level nodes, which are necessarily leaves. Each is in a two-leaf subtree labeled with a_i and a_{i+1}. Label such vertices' parent node (a_ia_{i+1}) and delete the largest level nodes (and the attached edges). Proceed inductively until one ends up with a parenthesization of $a_1 \cdots a_n$ at the root. □

22. Day 22

SOLUTION TO PROBLEM 25.1. Label each step in the path (starting from $(0,0)$) either E for east or N for north, and create the associated word of length $2n$ in the alphabet $\{E, N\}$. Now replace each E with a left parenthesis, and each N with a right parenthesis. In total, there are n opening and n closing parentheses, and the fact that the path never goes above the diagonal guarantees that at any given position in the string, there are at least as many opening as closing parentheses. As such we get n pairs of parentheses which are completely matched.

We claim that the set of n pairs of matched parentheses is in bijection with valid full parenthesizations of $n+1$ factors. We leave it the reader to decipher the following assignment and turn it into such a bijection:

$$(a(bc))d \mapsto ((a \cdot (b \cdot c)) \cdot d) \mapsto \cdot\cdot))\cdot) \mapsto (())().$$

□

SOLUTION TO PROBLEM 25.2. Both identities follow from algebra. □

23. Day 23

SOLUTION TO PROBLEM 26.2. These are the stars. Indeed, a star with i as its root and $\{0, 1, \ldots, n-1\} \setminus \{i\}$ as its leaves has Prüfer code i^{n-2} (by which we mean i repeated $n-2$ times). The converse clearly holds as well. □

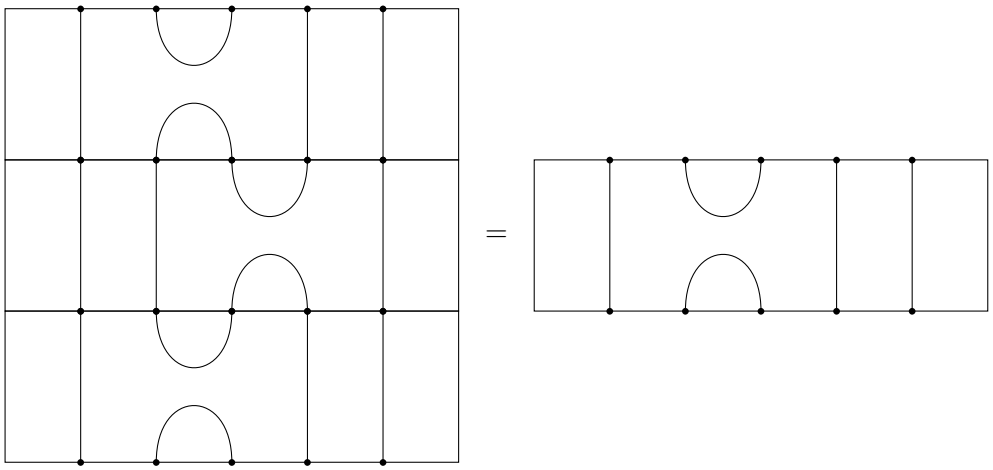
SOLUTION TO PROBLEM 26.3. These are the paths. Indeed, consider the path going from $\pi(0)$ to $\pi(1)$ to $\pi(2)$ to $\ldots$ to $\pi(n-1)$ where π is some permutation of $\{0, 1, \ldots, n-1\}$. At each step, the associated Prüfer code picks off one of the leaves, and these are all distinct values between 0 and $n-1$. It is easy to check the converse as well. □

24. Day 24

SOLUTION TO PROBLEM 27.1. Read the Temperley–Lieb diagram clockwise from the top left point. If a string is starting when you reach a point, draw record an open parenthesis; if a string is ending when you reach a point, record a closed parenthesis. This produces a function from Temperley–Lieb diagrams on $2n$ nodes to well-matched strings of n opening and n closing parentheses. We leave it to the reader to check that the function is a bijection. □

SOLUTION TO PROBLEM 27.2. (a) We can pull the vertical strings taught to recover the original diagram.

(b) Here we will record a picture of the second identity when $i = 2$ and $n = 5$. The others are similar.



(c) This is harder, but you can do it!

□

SOLUTION TO PROBLEM 27.3. (a) The trace of 1 is q^n.
(b) The trace of U_i is q^{n-1}.
(c) This is the problem of enumerating so-called *order n systems of meanders with k components*. It's an open question!

□

25. Day 25

SOLUTION TO PROBLEM 2.1. (a) The sample space is the collection of valid lottery tickets. If we assume that the lottery does not care about the order of the numbers, then we may model this sample space as $\binom{\underline{36}}{5}$, the collection of 5-element subsets of $\underline{36} = \{1, 2, \ldots, 36\}$.
(b) Sure! If the lottery is fair, then each ticket has an equal chance of being drawn.
(c) Suppose the winning ticket is the set $\{a_1, a_2, a_3, a_4, a_5\}$ where the a_i are distinct elements of $\underline{36}$. Then $B = \{t \in \binom{\underline{36}}{5} \mid a_i \notin t\}$. In other words, B is the collection of 5-element subsets of $\underline{36} \smallsetminus \{a_1, \ldots, a_5\}$. As such $|B| = \binom{31}{5}$ and $P(B) = \binom{31}{5}/\binom{36}{5} \approx 0.45$.
(d) We have $A \cap B = \varnothing$ and $A \cup B = \binom{\underline{36}}{5}$.
(e) It follows that $P(A) = P(A \cup B) - P(B) = 1 - \binom{31}{5}/\binom{36}{5} \approx 0.55$. If the lottery pays out 55% of the time, then it's not a very lucrative lottery for those running it!

□

SOLUTION TO PROBLEM 2.2. (a) We can number the cards 1 through 52, designating the ace of spades 1 and the king of hearts 2. An ordering of the cards corresponds to a permutation of $\underline{52}$, so the sample space is Σ_{52}, the set of permutations of $\underline{52}$. The event is

$$A = \{\pi \in \Sigma_{52} \mid \pi(1) < \pi(2)\}.$$

(b) We have $\Sigma_{52} \smallsetminus A = \{\pi \in \Sigma_{52} \mid \pi(2) < \pi(1)\}$. This is in bijection with A via the function that swaps the values of $\pi(1)$ and $\pi(2)$. Thus $|A| = |B|$, $A \cup B = \Sigma_{52}$, and $A \cap B = \varnothing$. As such,

$$1 = P(A \cup B) = P(A) + P(B) = 2P(A)$$

whence $P(A) = 1/2$.

□

TOWARDS A SOLUTION TO PROBLEM 2.3. This is a variant on the so-called *secretary problem*, née *fianceé problem*. We can use a *stopping rule* to increase our chance of winning: look at the first r cards and note the maximal value among them, M. For the subsequent $10 - r$ cards, select the first one larger than M. (If the tenth is not larger than M, select it and and bemoan your bad luck). With $r = 4$, you will select the largest number about 40% of the time, and this is the best r for 10 cards. A full analysis can be found in (Sardelis and Valahas, *Decision Making: A Golden Rule*, The American Mathematical Monthly Vol. 106, No. 3 (Mar., 1999), pp. 215-226). □

26. Day 26

SOLUTION TO PROBLEM 4.1. Since A and B are indpendent, we know that $P(A)P(B) = P(A \cap B)$. Since $A^c \cap B^c = (A \cup B)^c$, we aim to show that $P(A^c)P(B^c) = P((A \cup B)^c)$. We now compute

$$\begin{aligned} P(A^c)P(B^c) &= (1 - P(A))(1 - P(B)) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= 1 - P(A) - P(B) + P(A \cap B). \end{aligned}$$

We have $P(A) + P(B) - P(A \cap B) = P(A \cup B)$ (a probabilistic version of inclusion-exclusion) and thus

$$P(A^c)P(B^c) = 1 - P(A \cup B) = P((A \cup B)^c),$$

as desired. □

SOLUTION TO PROBLEM 4.2. First, we compute the probability of each event. Thinking of the coin flips as an n-bit binary sequence, we easily see that $P(A) = 2^{n-1}/2^n = 1/2$. Thinking of these sequences as subsets of an n-element set and recalling that there are the same number of even- and odd-sized subsets of $\underline{n}$, we get that $P(B) = 1/2$. Finally, $P(C) = \left(\sum_{k>n/2} \binom{n}{k}\right)/2^n$. When $n = 3$, we may compute this value to be $1/2$, and when $n = 4$ it is $5/16$.

The event $A \cap B$ consists of flip sequences with first flip a head and total heads even. This is the same as the first flip being heads and, amongst the subsequent $n - 1$ flips having an odd number of heads. There are 2^{n-2} such flip sequences, so $P(A \cap B) = 2^{n-2}/2^n = 1/4$ and A and B are independent as $P(A)P(B) = 1/2 \cdot 1/2 = 1/4$ as well.

The event $A \cap C$ consists of flip sequences with first flip a head and more heads than tails, total. When $n = 3$, $P(A \cap C) = 3/8 \neq 1/4$ so A and C are not independent in general.

The event $B \cap C$ consists of flip sequences with an even number of heads in which heads outnumber tails. When $n = 4$, that means there have to be 4 heads, so $P(B \cap C) = 1/16 \neq 1/2 \cdot 5/16$, so B and C are not independent in general. □

SOLUTION TO PROBLEM 4.3. (a) There is a $1/8$ probability of w winning against x, y, and z (think of this as three heads in a row). The event of w losing at least once against x, y, z is complementary and has probability $7/8$.

(b) Yes, player w's outcomes are independent of w''s.

(c) There are $n - 3$ players who are not x, y, or z. The probability that all of them lose against at least one of x, y, z is $(7/8)^{n-3}$.

(d) This event is the union over all 3-subsets $\{x, y, z\}$ of the event in (c). Thus its probability is at most $\binom{n}{3}(7/8)^{n-3}$ since $P(A \cup B) \leq P(A) + P(B)$ in general.

(e) If $\binom{n}{3}(7/8)^{n-3} < 1$, then in a positive fraction of tournaments, for each 3-subset of players there exists a player defeating all of them.

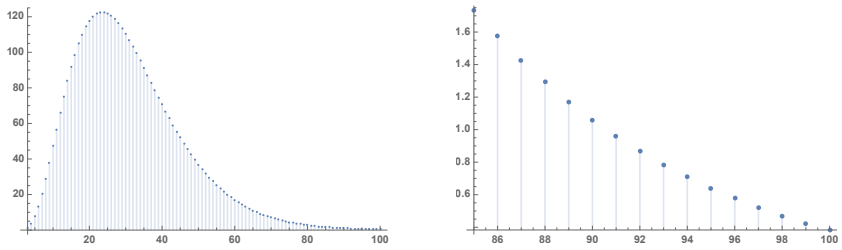


FIGURE 4. Plots of $\binom{n}{3}(7/8)^{n-3}$ with $3 \leq n \leq 100$ and $85 \leq n \leq 100$.

As it turns out, this expression is less than 1 for all $n \geq 91$, so we are certain that such a tournament exists whenever there are 91 or more players. You can see a plot of $\binom{n}{3}(7/8)^{n-3}$ in Figure 4.

□

27. Day 27

SOLUTION TO PROBLEM 6.1. By hypothesis, $P(A) = P(B) = P(C) = 1/3$. If we have initially picked A and the car is behind A, then the host will open B or C with equal probability. Thus $P(M_B|A) = 1/2, P(M_C|A) = 1/2, P(M_A|A) = 0$. If we have initially picked A and the car is behind B or C, then the host has only one door he can open, namely C or B, respectively. Thus $P(M_C|B) = 1$ and $P(M_B|C) = 1$.

Now suppose that the host opens door C. We want to compute $P(A|M_C)$ and $P(B|M_C)$. If the first is larger, we should stay; if the second is larger, we should switch; and if they are equal then it doesn't matter whether we stay or switch. By Bayes' Theorem and the Law of Total Probability,

$$\begin{aligned} P(A|M_C) &= \frac{P(M_C|A)P(A)}{P(M_C)} = \frac{1/2 \cdot 1/3}{P(M_C|A)P(A) + P(M_C|B)P(B) + P(M_C|C)P(C)} \\ &= \frac{1/6}{1/2 \cdot 1/3 + 1 \cdot 1/3 + 0 \cdot 1/3} = \frac{1/6}{1/2} \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} P(B|M_C) &= \frac{P(M_C|B)P(B)}{P(M_C)} = \frac{1 \cdot 1/3}{1/2} \\ &= \frac{2}{3}. \end{aligned}$$

The latter quantity is twice as large as the first, so we should switch! □

SOLUTION TO PROBLEM 6.2. Let K denote the event of knowing the answer to a particular problem and let M denote the event of correctly marking that problem. We want to determine $P(K|M)$, and do so with Bayes' Law. First note that the problem tells us that $P(K) = 3/5$, $P(M|K) = 1$, and $P(M|K^c) = 1/2$. (Here K^c is the complement of K, the event in which the student does not know the answer.) By the Law of Total Probability,

$$P(M) = P(M|K)P(K) + P(M|K^c)P(K^c) = 1 \cdot 3/5 + 1/2 \cdot 2/5 = \frac{4}{5}.$$

Thus

$$P(K|M) = \frac{P(M|K)P(K)}{P(M)} = \frac{1 \cdot 3/5}{4/5} = \frac{3}{4}.$$

In other words, there is a 75% chance of the student knowing the answer to a correctly marked question. □

28. Day 28

SOLUTION TO PROBLEM 8.1. (a) The potential values of the product are $2, 3, 4, 6, 8, 12$. For each such product, there is a unique 2-element subset $\{a, b\} \subseteq \{1, 2, 3, 4\}$ such that ab is the product in question. There are $\binom{4}{2} = 6$ such pairs, and thus each value has a $1/6$ probability of being chosen. We conclude that the expected value is $(2 + 3 + 4 + 6 + 8 + 12) \cdot 1/6 = 35/6 = 5.8333\ldots$.

(b) We have $\overline{AB} = 10A + B$ and $\overline{CD} = 10C + D$.

(c) It follows that $\overline{AB} \cdot \overline{CD} = (10A + B)(10C + D) = 100AC + 10AD + 10BC + BD$. By linearity of expectation,

$$E(\overline{AB} \cdot \overline{CD}) = (100 + 10 + 10 + 1) \cdot \frac{35}{6} = \frac{4235}{6} = 705.8333\ldots.$$

□

SOLUTION TO PROBLEM 8.2. (a) We have $T = X_0 + X_1 + \cdots + X_{n-1}$.

(b) We are seeking to collect one of the $n - k$ uncollected coupons out of the n total coupons, so $p_k = \frac{n-k}{n}$ and $E(X_k) = \frac{1}{p_k} = \frac{n}{n-k}$.

(c) By linearity of expectation,

$$\begin{aligned} E(T) &= \sum_{k=0}^{n-1} E(X_k) \\ &= \sum_{k=0}^{n-1} \frac{n}{n-k} \\ &= n \sum_{k=0}^{n-1} \frac{1}{n-k} \\ &= n \sum_{i=1}^{n} \frac{1}{i}. \end{aligned}$$

(d) It is beyond the scope of this course to prove so, but $E(T) = n \log n + \gamma n + O(1/n)$ where $\gamma \approx 0.577$ is the *Euler-Mascheroni constant*.

□

29. Day 29

SOLUTION TO PROBLEM 10.1. We can model the sample space as $\underline{6} \times \underline{6}$, in which case the event of doubles is the diagonal $\Delta = \{(a, a) \mid a \in \underline{6}\}$. Then under the uniform distribution, $P(\Delta) = 6/36 = 1/6$. Let X be the number of doubles out of 12 rolls. Let I_j denote teh indicator variable for the j-th roll being a double. Then $E(I_j) = P(I_j = 1) = P(\Delta) = 1/6$. Since $X = I_1 + \cdots + I_{12}$, $E(X) = 12 \cdot 1/6 = 2$. We expect two doubles to be rolled. □

SOLUTION TO PROBLEM 10.2. Let X be the number of people who show up for the flight. We are looking for $P(X > 200) = P(X = 201) + P(X = 202) + \cdots + P(X = 205)$. Since this is a binomial random variable, $P(X = k) = \binom{205}{k}(0.95)^k(0.05)^{205-k}$. Thus

$$P(X > 200) = \sum_{k=201}^{205} \binom{205}{k}(0.95)^k(0.05)^{205-k} \approx 0.02236.$$

We conclude that the flight will be oversold about 2.2% of the time. □

SOLUTION TO PROBLEM 10.3. This is a geometric random variable with $p = P(X > 200) \approx 0.02236$. As such, the expected number of flights until an oversold one is $1/p \approx 44.7$. □

30. Day 30

SOLUTION TO QUESTION 1.1. Since $b = 1 \cdot b$ for all $b \in \mathbb{Z}$, we have always have $1 \mid b$. Similarly, $b = (-1) \cdot (-b)$, so $-1 \mid b$ for all $b \in \mathbb{Z}$. Since $0 = a \cdot 0$, we always have $a \mid 0$, and sicne $a = a \cdot 1$, we always have $a \mid a$. □

SOLUTION TO PROBLEM 1.2. By hypothesis, there are integers m, m' such that $b = am$ and $c = bm'$. Thus $c = (am)m' = a(mm')$. Since mm' is an integer, this tells us that $a \mid c$. □

SOLUTION TO QUESTION 1.3. Since $9 = 3 \cdot 3$, it goes above 3 with lines coming in from 1 and 3, and lines going up to all multiples of 9. □

SOLUTION TO PROBLEM 1.4. By hypothesis, $b = ak$ and $c = a\ell$ for some integers k, ℓ. Thus $mb + nc = mak + na\ell = a(mk + n\ell)$, and since $mk + n\ell \in \mathbb{Z}$ we have that $a \mid mb + nc$. □

SOLUTION TO QUESTION 1.5. Above 1 and below everything else. □

SOLUTION TO PROBLEM 1.6. First suppose that n is prime. Then it is not divisible by any positive integer except 1 and n, and thus is not divisible by the prime numbers in question.

Now suppose that n is not prime, which case it has prime factorization $n = p_1 p_2 \cdots p_k$ with $p_1 \leq \cdots \leq p_k$ all prime. Suppose for contradiction that $\sqrt{n} < p_1$. Then $n = \sqrt{n} \cdot \sqrt{n} < p_1 p_2 \leq n$, *i.e.*, $n < n$, a contradiction. □

SOLUTION TO PROBLEM 1.7. The divisors of n take the form $p_1^{b_1} \cdots p_k^{b_k}$ with $0 \leq b_i \leq a_i$. Since there are $a_i + 1$ potential values of b_i, we know that n has $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ divisors. □

SOLUTION TO PROBLEM 1.8. We want to show that every integer $n \geq 2$ has a prime factorization. Since 2 is prime, the base case holds. Fix an integer $n \geq 2$ and suppose that all integers $2 \leq m \leq n$ have prime factorization. If $n + 1$ is prime, then it has a prime factorization (itself), so suppose $n + 1$ is composite. Then there are integers $2 \leq a, b \leq n$ such that $n + 1 = ab$. By the strong inductive hypothesis, both a and b have prime factorizations, and the product of those factorizations is in turn a prime factorization of $ab = n + 1$. □

31. Day 31

SOLUTION TO PROBLEM 2.1. If a divides n and $n + 1$, then a divides $(n + 1) - n = 1$. The only positive divisor of 1 is 1. □

SOLUTION TO QUESTION 2.2. Start with $n = 2$ so that $N_2 = 2 \cdot 3 = 6$, $N_3 = 6 \cdot 7 = 42$, $N_4 = 42 \cdot 43 = 1806$, and $N_5 = 1806 \cdot 1807 = 3,263,442$. The smallest number with 5 distinct prime divisors is $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$, so this is not very efficient! □

SOLUTION TO PROBLEM 2.3. By the prime number theorem, $\pi(n)\log n/n \to 1$ as $n \to \infty$. Since $\log n \to \infty$, we must have $\pi(n)/n \to 0$ (otherwise PNT would not hold).

Now fix $a \in \mathbb{R}$ and observe that

$$\frac{\pi(n)(\log n - a)}{n} = \frac{\pi(n)\log n}{n} - \frac{a\pi(n)}{n}.$$

By PNT, the first term goes to 1, and we have just proven that the second term goes to 0. Thus $\pi(n) \sim n/(\log n - a)$. □

32. Day 32

SOLUTION TO PROBLEM 3.1. (a) If we allow all colorings with each spoke one of a colors, then there are a^p colorings. Of these, a colorings are monochromatic, so there are $a^p - a$ non-monochromatic colorings.

(b) The phenomenon is generic when the number of spokes is prime. Indeed, if we can rotate by $2\pi k/p$ (for $1 \le k < p$ an integer) and get the same coloring, then the pattern repeats every k spokes, and thus k divides p. Since p is prime, $k = 1$, but that means the pattern is monochromatic.

(c) The nailed-to-the-wall count of $a^p - a$ overcounts by a factor of p (the number of ways to rotate one pattern into others). Thus $\frac{a^p-a}{p}$ is an integer; in particular, p divides $a^p - a$. This is *Fermat's little theorem*.

□

COMMENTS ON PROBLEM 3.2. For $a = 2, 3, 4, 5$, the values are 12, 127, 696, and 2,630, respectively. These are difficult counting problems that require a lot of care with the symmetries involved. The more general problem of n spoke wheels with a colors is called the *combinatorial necklace problem*. (It is more traditional to phrase the problem in terms of necklaces and colored beads instead of wagon wheels and spokes.) A nice illustration of combinatorial necklaces and links to the relevant mathematics is available at https://www.jasondavies.com/necklaces/. □

33. Day 33

SOLUTION TO PROBLEM 4.1. The gcd is the "greatest lower bound" (or *infimum*) of the common divisors of 84 and 105. □

SOLUTION TO PROBLEM 4.2. The equation $r_{n-1} = q_n r_n$ clearly exhibits that $r_n \mid r_{n-1}$. Fix $0 \le k \le n$ and assume for (downward, strong) induction that $r_n \mid r_\ell$ for $k \le \ell \le n$. The equation $r_{k-1} = q_k r_k + r_{k+1}$ expresses r_{k-1} as an integral linear combination of r_k and r_{k+1}, both of which are divisible by r_n, hence r_n divides r_{k-1} as well. We conclude that $r_n \mid r_k$ for all $-1 \le k \le n$, including $r_{-1} = a$ and $r_0 = b$.

Beginning with $a = q_0 b + r_1$, we have $r_1 = a - q_0 b$ and hence any common divisor of a and b divides r_1. In general, $r_k = r_{k-2} - q_{k-1} r_{k-1}$, permitting a strong inductive proof that $\gcd(a, b)$ divides r_k for $-1 \le k \le n$.

We now know that r_n is a common divisor of a and b and that $\gcd(a, b) \mid r_n$. This makes r_n a divisor of a and b which is at least as large as $\gcd(a, b)$, whence $r_n = \gcd(a, b)$. □

SOLUTION TO PROBLEM 4.3. The Euclidean algorithm runs as follows:

$$\begin{aligned} 23 &= 1 \cdot 13 + 10 \\ 13 &= 1 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0. \end{aligned}$$

This corresponds to breaking a 23×13 rectangle into one 13×13 square, one 10×10 square, three 3×3 squares, and three 1×1 squares.

If you start with F_{n+1} and F_n, the Euclidean algorithm has $q_k = 1$ for all k and you get the Fibonacci approximation to the golden rectangle. □

SOLUTION TO PROBLEM 4.4. The Euclidean algorithm runs as follows:

$$\begin{aligned} 45 &= 2 \cdot 16 + 13 \\ 16 &= 1 \cdot 13 + 3 \\ 13 &= 4 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0. \end{aligned}$$

We have $x_k = q_{k-1}$. □

34. Day 34

SOLUTION TO QUESTION 5.1. We have $a \equiv b \pmod 2$ when a and b are both odd or both even. Since $1 \mid a - b$ for all a, b, we always have $a \equiv b \pmod 1$. We only have $0 \mid a - b$ when $a - b = 0$, i.e., congruence modulo 0 is just equality of integers. □

SOLUTION TO PROBLEM 5.2. Fix m and write $\equiv$ for congruence modulo m. This relation is reflexive ($a \equiv a$) since $m \mid 0 = a - a$. It is symmetric since when $m \mid a - b$ we also have $m \mid b - a = (-1)(a - b)$. For transitivity, suppose $a \equiv b$ and $b \equiv c$, in which case there are integers k, ℓ such that $a - b = km$ and $b - c = \ell m$. Then $a - c = (a - b) + (b - c) = (k + \ell)m$, so $a \equiv c$, as desired.

Write $\overline{a}$ for the equivalence class of a modulo m. Then

$$\overline{a} = \{a + km \mid k \in \mathbb{Z}\} = a + m\mathbb{Z}$$

and there are exactly m equivalence classes,

$$\overline{0}, \overline{1}, \ldots, \overline{m-1}.$$

□

SOLUTION TO PROBLEM 5.3. We define $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{a} \cdot \overline{b} = \overline{ab}$. These are well-defined operations since

$$\overline{(a + km) + (b + \ell m)} = \overline{(a + b) + (k + \ell)m} = \overline{a + b}$$

and

$$\overline{(a + km)(b + \ell m)} = \overline{ab + (a\ell + bk + k\ell m)m} = \overline{ab}.$$

Since $\overline{a} + \overline{m - a} = \overline{m} = \overline{0}$, $\mathbb{Z}/m\mathbb{Z}$ has additive inverses. □

SOLUTION TO PROBLEM 5.5. The graph $G(\overline{a}, m)$ consists of disjoint directed cycles, all of the same size. Each cycle has length ℓ where ℓ is the smallest positive integer such that $\ell a \equiv 0 \pmod m$. We can re-express this number as $\ell = \operatorname{lcm}(a, m)/a$. □

35. Day 35

SOLUTION TO PROBLEM 6.1. (a) The congruence $a^p \equiv a \pmod p$ means that p divides $a^p - a$, as desired.

(b) If $a \not\equiv 0 \pmod p$, then a has a multiplicative inverse modulo p. Multiplying both sides of the congruence by this inverse results in $a^{p-1} \equiv 1 \pmod p$.

(c) Working in $\mathbb{Z}/p\mathbb{Z}$ (and dropping the bars from our notation), let $b = a^{(p-1)/2}$. Then $b^2 = a^{p-1} = 1$, whence $0 = b^2 - 1 = (b + 1)(b - 1)$. Thus $b = \pm 1$.

(d) First let $o = o_p(a)$ and use the division algorithm to write $p-1 = qo+r$ where $0 \le r < o$. Then $qo = p-1-r$ and thus $1 = 1^q = (a^o)^q = a^{qo} = a^{p-1-r}$. Multiplying by a^r we get $a^r = a^{p-1} = 1$. Since o is the minimal positive integer such that $a^o = 1$, we know that $r = 0$, whence $o \mid p-1$, as desired.

(e) We will leave this is a challenge problem — it's hard, but important!

(f) Take $1 \le m \le n \le p-1$ and suppose $a^m = a^n \in \mathbb{Z}/p\mathbb{Z}$. Then $1 = a^{n-m}$ where $0 \le n-m \le p-2$. Since $o_p(a) = p-1$, we must have $n - m = 0$, *i.e.*, $n = m$. Since the values of a^n with $1 \le n \le p-1$ are distinct, there are $p-1$ of them, and they all live in $\mathbb{Z}/p\mathbb{Z} \smallsetminus \{0\}$ (which has size $p-1$), we get that $\mathbb{Z}/p\mathbb{Z} \smallsetminus \{0\} = \{a^n \mid 1 \le n \le p-1\}$.

□

SOLUTION TO PROBLEM 6.2. This works in general because with fixed $a, c \in \mathbb{Z}/p\mathbb{Z}^\times$, there is a unique $b \in \mathbb{Z}/p\mathbb{Z}^\times$ such that $ab = c$. (Indeed, $b = c/a$.) □

SOLUTION TO PROBLEM 6.3. The squaring function is 2-to-1 onto its image, so its image must have size $(p-1)/2$. Thus $1/2$ of the equations $x^2 \equiv a \pmod p$ have solutions (for varying $1 \le a \le p-1$).

The final observation is just that $1^2 = 1$ and $(p-1)^2 = (-1)^2 = 1$. □

SOLUTION TO PROBLEM 6.4. Number the days 0 through 6, starting with Sunday, and note that Thursday corresponds to 4. You take the n-th dose on the day corresponding to the congruence class of $5(n-1)$ modulo 7. Thus we are looking for the minimum $n \ge 50$ such that $5(n-1) \equiv 4 \pmod 7$. Adding 5 to both sides, this becomes $5n \equiv 2 \pmod 7$. The multiplicative inverse of 5 mod 7 is 3 (since $3 \cdot 5 = 15 \equiv 1 \pmod 7$), and thus $n \equiv 6 \pmod 7$. Recalling that $50 \equiv 1 \pmod 7$, we see that n must be 55. □

36. Day 36

SOLUTION TO PROBLEM 7.1. (a) Let's first consider the complementary event of $r \in \underline{n}$ divisible by p_i. These are precisely $p_i, 2p_i, 3p_i, \ldots, (n/p_i)\cdot p_i$, so there are n/p_i such integers. As such, $|ND_i| = n - n/p_i$ and

$$P(ND_i) = \frac{n - n/p_i}{n} = 1 - \frac{1}{p_i}.$$

(b) In order that $\gcd(r, n) = 1$, r and n must share no common divisors. This is the case if and only if $p_i \nmid r$ for all prime divisors p_i of n. This in turn is the intersection $ND_1 \cap \cdots \cap ND_k$.

(c) These events are independent if and only if their complements are independent. (Check this!) A number is divisible by $p_1, \ldots, p_k$ if and only if it is divisible by $p_1 \cdots p_k$. The probability of the latter event is

$$\frac{n/(p_1 \cdots p_k)}{n} = \frac{1}{p_1 \cdots p_k}.$$

This is equal to

$$\frac{1}{p_1} \cdots \frac{1}{p_k},$$

the product of the individual events. This proves independence.[5]

□

[5]For full independence, we would need to check this for any subset of prime divisors, but the argument is the same.

37. Day 37

SOLUTION TO QUESTION 9.1. First multiply the first congruence by the mod 7 inverse of 2, which is 4, to get $x \equiv 6 \pmod 7$. Then multiply the second congruence by the mod 8 inverse of 3, which is 3, to get $x \equiv 4 \pmod 8$.

Since 7 and 8 are relatively prime, Sunzi's theorem applies, there is exactly one solution $0 \leq x_0 < 7 \cdot 8 = 56$ and all other solutions are of the form $x_0 + 56n$ for some $n \in \mathbb{Z}$. The solutions to $x \equiv 4 \pmod 8$ between 0 and 55 are

$$4, 12, 20, 28, 36, 44, 52.$$

The only one of these satisfying $x \equiv 6 \pmod 7$ is $x_0 = 20$. Thus all solutions are of the form $20 + 56n$, $n \in \mathbb{Z}$. □

SOLUTION TO PROBLEM 9.2. The remainder under consideration is the unique r such that $0 \leq r < 1728$ and $r \equiv 135^3 \pmod{1728}$. Such an r also satisfies the congruences

$$r \equiv 135^3 \pmod{64}$$
$$r \equiv 135^3 \pmod{27}.$$

Since $135 \equiv 7 \pmod{64}$, we know that $135^2 \equiv 7^2 \equiv -15 \pmod{64}$ and $135^3 \equiv -15 \cdot 7 \equiv -105 \equiv 23 \pmod{64}$. Similarly, since $135 \equiv 0 \pmod{27}$, we have $135^3 \equiv 0 \pmod{27}$. Thus we may rewrite the system of congruences as

$$r \equiv 23 \pmod{64}$$
$$r \equiv 0 \pmod{27}.$$

By the second congruence, we know that r is of the form $27k$ for some integer k. Since $\gcd(27, 64) = 1$, we know that 27 has a multiplicative inverse mod 64. Running the extended Euclidean algorithm, we find that 19 is its inverse, whence $k \equiv 19 \cdot 23 \equiv 437 \equiv 53 \pmod{64}$. Thus $r = 27 \cdot 53 = 1431$. □

SOLUTION TO PROBLEM 9.3. The extension is possible and hinges on considering congruence classes modulo $p_i^{a_i}$. For variety's sake, here is a totally different method:

Enumerate the integers between 0 and n which are relatively prime to n: $x_1, x_2, \ldots, x_{\phi(n)}$. If $ax_i \equiv ax_j \pmod n$, then, multiplying by a^{-1} mod n gives $x_i \equiv x_j \pmod n$. This means that multiplication by a permutes the x_i's. As such,

$$\prod_{i=1}^{\phi(n)} x_i \equiv \prod_{i=1}^{\phi(n)} ax_i \equiv a^{\phi(n)} \prod_{i=1}^{\phi(n)} x_i \pmod n.$$

Multiplying by $(\prod x_i)^{-1}$ mod n gives $1 \equiv a^{\phi(n)} \pmod n$, as desired. □