MATH 113: DISCRETE STRUCTURES
THE FIBONACCI SEQUENCE VIA GENERATING FUNCTIONS

Suppose $a_0, a_1, a_2, \ldots$ is a sequence of combinatorial interest. A standard tool in combinatorics is to write down the corresponding *generating function*:

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

For instance, the generating function for the Fibonacci sequence is

$$F(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + \cdots .$$

Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ are generating functions. We can add and multiply them just as we would ordinary (finite) polynomials. The coefficient of x^n in $f(x) + g(x)$ is $a_n + b_n$, and the coefficient of x^n in $f(x)g(x)$ is

$$a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Note that determining the coefficient of x^n is a finite process, and we do not need to worry about convergence of infinite sums to make sense of these operations.

Here is an example computation involving $1 - x$ and $1 + x + x^2 + x^3 + \cdots$:

$$\begin{aligned}(1-x)(1 + x + x^2 + x^3 + \cdots) &= 1 - x + x - x^2 + x^2 - x^3 + x^3 - x^4 + x^4 - \cdots \\ &= 1.\end{aligned}$$

Therefore

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{1}$$

in the world of generating functions. (Since we are not setting x equal to a particular number, we don't need to worry about convergence.)

Let's return to the Fibonacci generating function

$$F(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots + F_n x^n + \cdots .$$

We have

$$xF(x) = x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + \cdots + F_{n-1} x^n + \cdots$$

and

$$x^2 F(x) = x^3 + x^4 + 2x^5 + 3x^6 + 5x^7 + 8x^8 + \cdots + F_{n-2} x^n + \cdots .$$

A little algebra and the Fibonacci recurrence reveals that

$$F(x) - x = xF(x) + x^2 F(x),$$

which can be rewritten as

$$F(x) - xF(x) - x^2 F(x) = x$$

Equivalently,

$$F(x)(1 - x - x^2) = x,$$

i.e.,

$$F(x) = \frac{x}{1 - x - x^2}.$$

We now define

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \overline{\phi} = \frac{1-\sqrt{5}}{2}.$$

The reader may verify that

$$1 - x - x^2 = (1-\phi x)(1-\overline{\phi}x).$$

From this, it follows that

$$\begin{aligned} F(x) &= \frac{x}{1-x-x^2} \\ &= \frac{x}{(1-\phi x)(1-\overline{\phi}x)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x} - \frac{1}{1-\overline{\phi}x}\right) \end{aligned}$$

By (1) with ϕx in place of x,

$$\frac{1}{1-\phi x} = 1 + \phi x + (\phi x)^2 + (\phi x)^3 + \cdots = 1 + \phi x + \phi^2 x^2 + \phi^3 x^3 + \cdots.$$

Similarly,

$$\frac{1}{1-\overline{\phi} x} = 1 + \overline{\phi} x + (\overline{\phi} x)^2 + (\overline{\phi} x)^3 + \cdots = 1 + \overline{\phi} x + \overline{\phi}^2 x^2 + \overline{\phi}^3 x^3 + \cdots.$$

We conclude that

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x} - \frac{1}{1-\overline{\phi}x}\right) \\ &= \frac{1}{\sqrt{5}}\Big((1 + \phi x + \phi^2 x^2 + \phi^3 x^3 + \cdots) - (1 + \overline{\phi} x + \overline{\phi}^2 x^2 + \overline{\phi}^3 x^3 + \cdots)\Big) \\ &= \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}(\phi^k - \overline{\phi}^k)x^k. \end{aligned}$$

In order for this identity to hold, we must have the coefficient of each x^n match. For the left-hand side, this coefficient is F_n by definition. For the right-hand side, this coefficient is $\frac{1}{\sqrt{5}}(\phi^n - \overline{\phi}^n)$. We have thus proven that

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).}$$

If you are interested in learning more about generating functions, the book *Generatingfunctionology* by Herbert Will is highly recommended. It is available for free at the following link:

`https://www.math.upenn.edu/~wilf/DownldGF.html`.