Challenge problems for Math 113

Rewards!

- For each solution, I will reward five points towards your score on the homework.
- If I receive solutions to at least five distinct problems, I will make a treat for the class (*e.g.*, cookies or brownies). This will repeat every time we accumulate another five solutions.

Rules!

- **You are not allowed to use the internet.** (What's the fun in that—and it's often surprisingly easy to tell!)
- To receive points for homework, you solution must be completely your own. If you prefer working with others—that's fine. In that case turn in a jointly-authored solution. This will count towards the class treat.
- Solutions must be completely correct and easy to read. If they are not typeset in TeX, they need to be extremely neat with good handwriting with nice spacing. Complete sentences are mandatory.
- If your solution is convoluted with lots of hard-to-follow logic, I reserve the right to send it back without looking at it.

PROBLEMS (This list will grow throughout the semester.)

1. Can one place eight knights on a standard 8×8 chess board such that every square (occupied or not), is attacked by a knight?
2. For a finite set S of positive integers, let $a(S)$ be its alternating sum, which is defined in the following way. Write $S = \{s_1 > s_2 > \cdots > s_r\}$, and let
$$a(S) := s_1 - s_2 + s_3 - s_4 + \cdots + (-1)^{r-1} s_r.$$
Compute the sum of $a(S)$ over all subsets $S \subseteq \{1, \ldots, n\}$.
3. In how many ways can we place some number of checkers on a standard chess board such that each row and column has an even number of checkers?
4. You are given a number x, and you are allowed to perform two operations on it; at each step, you can change it either $\lfloor x/2 \rfloor$ or to $4x + 1$. Starting with 0, how many integers less than or equal to 2047 is it possible to get using an arbitrary sequence of operations?
5. Prove
$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$
without using algebra, i.e., with a combinatorial proof.

6. There are n people with cameras wandering around on a field. At noon, each person takes a photo of whoever is closest. Prove that if n is odd, at least one person does not have their photo taken. (Assume that at noon there are no ties for closest person.)

7. A plane is divided into regions by a finite number of lines. Prove that it is possible to color each of the resulting regions red or blue so that any two bordering regions are of opposite colors. (Two regions border if and only if their common boundary is a line segment; in particular, regions that only meet at a single point do not border.)

8. Fix some integer $n \geq 2$. Given a graph on $2n$ vertices, suppose we draw some $n^2 + 1$ edges. Prove that we drew a triangle.

9. Let S be a set with 100 elements, and let N be an integer with $0 \leq N \leq 2^{100}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:

   (a) the union of any two white subsets is white;

   (b) the union of any two black subsets is black;

   (c) there are exactly N white subsets.

10. Prove that for every non-negative integer n, the integers $1, 2, \ldots, 2^{n+1}$ can be partitioned into two sets A and B such that for all $0 \leq i \leq n$,

$$\sum_{x \in A} x^i = \sum_{x \in B} x^i.$$