MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 6

Question 1. How many permutations of an n-element set have exactly one fixed point? Take an integer k such that $1 \leq k \leq n$; how many permutations of an n element set have exactly k fixed points?

Question 2. Recall that we have defined $n¡$ to be the number of derangements of $\underline{n}$ (the number of permutations of $\{1, \ldots, n\}$ with no fixed points). Consider the following inductive "proof" that $n¡ = (n-1)!$ for all $n \geq 2$:

> For $n = 2$ the formula holds, so take some $n \geq 2$ and assume that $n¡ = (n-1)!$. Let π be a permutation of $\{1, 2, \ldots, n\}$ with no fixed points. We want to extend it to a permutation π' of $\{1, 2, \ldots, n+1\}$ with no fixed points. We choose a number $i \in \{1, 2, \ldots, n\}$, and we define $\pi'(n+1) = \pi(i)$, $\pi'(i) = n+1$, and $\pi'(j) = \pi(j)$ for $j \neq i, n+1$. This defines a permutation of $\{1, 2, \ldots, n+1\}$, and it's easy to check that it has no fixed points. For each of the $n¡ = (n-1)!$ possible choices of π, the index i can be chosen in n ways; therefore, $(n+1)¡ = (n-1)! \cdot n = n!$.

This "proof" gives a procedure for producing n derangements of $\underline{n+1}$ from each derangement of $\underline{n}$. Apply this procedure to all of the derangements of $\underline{3} := \{1, 2, 3\}$, and use your result to reveal the error. For ease of notation, you can represent a permutation π by its list of values $\pi(1)\pi(2)\ldots\pi(n)$. For instance, the permutation of $\underline{2}$ defined by $\pi(1) = 2$ and $\pi(2) = 1$ could be denoted by 21.