MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 5

Problem 1. We would like to make a committee out of n people. By definition, a committee is a collection of the n people subject only to the condition that each committee must have a unique chairperson. The same set of people but with a different person being chair counts as a different committee. Using this context, answer the following questions to give a combinatorial proof of the identity

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

(1) What sequence of choices leads to the count of the number of committees given by the right-hand side, $n2^{n-1}$?
(2) What sequence of choices leads to the count of the number of committees given by the left-hand side, $\sum_{k=1}^{n} k\binom{n}{k}$?

Problem 2. Let X be set of all subsets of size three from $\{1, \ldots, n+2\}$. For instance, if $n = 2$ we would have

$$X = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

In general, the number of such subsets is $|X| = \binom{n+2}{3}$. Each element of X consists of three numbers, which we list in order: $a < b < c$. For each integer b, let X_b be all subsets of $\{1, \ldots, n+2\}$ of the form $\{a, b, c\}$ for which $a < b < c$. We get a partition of X:

$$X = X_2 \amalg X_3 \amalg \cdots \amalg X_{n+1},$$

and hence.

$$(\star) \qquad |X| = |X_2| + |X_3| + \cdots + |X_{n+1}|.$$

(1) Determine (with explanation, of course) the size $|X_b|$ for $n = 2, 3, \ldots, n+1$ in terms of b and n.
(2) Equation $(\star)$ becomes what identity? (Note: to be sure of your answer, you should check it for small n on scratch paper.)

Note. Combinatorial identities often arise from partitioning a set. On your own, you may want to consider how the Problem 1 involves a partition.