MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 4

Suppose we have an identity $E = F$ where E and F are two algebraic expressions that evaluate to the same integer (see the examples below). A *combinatorial* explanation for the identity $E = F$ requires identifying both E and F as solutions to counting problems and explaining why these counting problems should have the same solution. As an example, we give a proof of the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

in the case where $n > k > 0$. (It is true for general n and k, but we will skip these trivial cases.)

Proof. Let $S = \{1, \dots, n\}$. The left-hand side counts the k-subsets of S. Each k-subset of S is of exactly one of two types: (1) those that contain n, and (2) those that do not. To find the number of k-subsets of S, we can just count the numbers of each type and add. A subset of size k containing n, i.e, of type (1), is the same thing as a subset of $\{1, \dots, n-1\}$ of size $k-1$ to which we then append n. Thus, there are $\binom{n-1}{k-1}$ subsets of type (1). A k-subset of S that does not contain n, i.e., of type (2), is the same as a subset of $\{1, \dots, n-1\}$, and there are $\binom{n-1}{k}$ of these. $\square$

Problem 1. Consider the following identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) Give an algebraic proof of the identity using the formula $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. Try to give your proof in the form

$$\begin{aligned} k\binom{n}{k} &= blah \\ &= blah \\ &\vdots \\ &= n\binom{n-1}{k-1}. \end{aligned}$$

(b) Suppose we have a set S of n people, and we want to choose a committee of k people, one of whom is designated the chair of the committee. Someone asks us to count the number of such "chaired committees". Give a combinatorial explanation of the identity by explaning why each side provides the count.

Problem 2. Give a combinatorial explanation of the following identity:

$$\binom{17}{5} = \binom{10}{0}\binom{7}{5} + \binom{10}{1}\binom{7}{4} + \binom{10}{2}\binom{7}{3} + \binom{10}{3}\binom{7}{2} + \binom{10}{4}\binom{7}{1} + \binom{10}{5}\binom{7}{0}.$$

Hint: you might think about coloring the elements of a set.