MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 2

For full credit: your solutions must adhere to the templates and advice given on pages 2–3. Please read carefully.

Problem 1. Let $f : A \to B$ be a function. Show that a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ exists if and only if f is surjective.

Problem 2. Suppose that $f\colon A \to B$ and $g\colon B \to C$ are composable functions.
(a) If $g \circ f$ is surjective, does g have to be surjective? Does f have to be surjective?
(b) If $g \circ f$ is injective, does g have to be injective? Does f have to be injective?

PROOF-WRITING TEMPLATES/ADVICE

The exact form the statement of a theorem takes almost always sets up expectations in a mathematician's mind for the form of its proof. Not meeting that expectation often leads to confusion (unless safeguards are otherwise taken by the proof-writer). Below, I will try to make some of those forms explicit by providing templates, and you should try to adhere to them in your own writing.

In the following, the symbols P and Q denote mathematical statements that may be true or false.

I. Implications.

Theorem 3. *If P, then Q.*

Proof. Suppose P. Then blah, blah, blah, ...
....
....
It follows that Q. □

Theorem 4. *P if and only if Q.*

Proof. ($\Rightarrow$) Suppose P. Then blah, blah, blah, ...
....
....
It follows that Q.

($\Leftarrow$) Now suppose Q. Then blah, blah, blah, ...
....
....
It follows that P. □

II. Proving injectivity/surjectivity/bijectivity.

Theorem 5. *The function $f\colon A \to B$ is injective.*

Proof. Let $x, y \in A$, and suppose that $f(x) = f(y)$. Then blah, blah, blah. It follows that $x = y$. Hence, f is injective. □

Theorem 6. *The function $f\colon A \to B$ is surjective.*

Proof. Let $b \in B$. Then blah, blah, blah. Thus, there exists $a \in A$ such that $f(a) = b$. Hence, f is surjective. □

Theorem 7. *The function $f\colon A \to B$ is bijective.*

Proof. (Alternative 1.) We first show that f is injective. [Follow the template above to show injectivity.] Next, we show f is surjective. [Follow the template above toe show surjectivity.] □

Proof. (Alternative 2) Define $g\colon B \to A$ as follows: blah, blah, blah. Note that $g \circ f = \mathrm{id}_A$ since blah, blah, blah. Next, note that $f \circ g = \mathrm{id}_B$ since blah, blah, blah. □

III. Use of examples.

Theorem 8. *Show that P does not imply Q.*

Proof. [Give the simplest and most concrete example of a case in which P is true and Q is not. If you were, instead, trying to show P implies Q, then an example might be useful but would not suffice as a proof. However, in trying to show P does not imply Q, it is the opposite: you are obliged to provide an example, and a general explanation might be useful but does not suffice.] □

IV. Proof by contrapositive or contradiction.

Instead of proving P implies Q directly, it is sometime tempting to prove the logically equivalent statement: "if not Q, then not P". Similarly, it is often tempting to start off by assuming that P is true and Q is not true and then showing that a contradiction arises. Advice: *whenever* you take that route with a proof, when you are done, go back and see if you can prove P implies Q directly. More often than not, the direct proof will be cleaner and the indirect one, involving negation, is convoluted in comparison.

As an example, consider a theorem of the form: "if the function f is defined by blah, then f is injective". To prove this you could suppose that $f(x) = f(y)$ and $x \neq y$, i.e., f is not injective, and then argue that you get a contradiction. This would be reasonable since this is the way we store the notion of injectivity in our brains: two different elements in the domain are not sent by the function to the same element in its codomain. However, it is almost always the case that a direct proof, as given in the template, above, is more clear: assume $f(x) = f(y)$ and show that this means $x = y$. (In fact, this latter proof exactly expresses the definition of injectivity given in our course notes.)