Math 113 Group Problems for Monday, Week 13

PROBLEM 1.

(a) Factor 336 and use the factorization to compute $\phi(336)$, i.e., the number of positive integers a less than 336 such that $\gcd(a, 336) = 1$.

(b) What is the remainder of $5^{960000290}$ upon division by 336?

PROBLEM 2. (Sketch of probabilistic proof of Euler's formula for the totient function.) Let $n = p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of the positive integer n. Let $\underline{n} := \{1, \dots, n\}$ be our sample space with uniform distribution. For $i = 1, \dots, k$, define the event E_i to be the set of $r \in \underline{n}$ such that $p_i \nmid r$.

(a) What are the sets E_i in the case $n = 60$? What are the probabilities $P(E_i)$.

(b) Back to the case of general n, what is $P(E_i)$ for each i?

(c) Let R be the collection of $r \in \underline{n}$ which are relatively prime to n. Check that $R = E_1 \cap E_2 \cap \cdots \cap E_k$.

(d) It turns out that $P(R) = P(E_1) \cdots P(E_k)$. Use this fact to prove that

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

PROBLEM 3. For each $k \in \{1, 2, 3, 4\}$, find all numbers n such that $\phi(n) = k$.

PROBLEM 4. How does Euler's formula show that if $\gcd(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$? Find the smallest integers a and b such that $\phi(ab) \neq \phi(a)\phi(b)$.

PROBLEM 5. Describe the positive integers n for which $\phi(n) | n$.