Math 113 Group Problems for Friday, Week 10

Problem 1.

(a) Roll a pair of dice twelve times. If you don't have dice immediately on hand, you can go to https://www.random.org/dice/?num=2. Record the first roll on which you roll doubles and also the total number of doubles that you roll and report these numbers to the instructor.

SOLUTION: My results where:

$$(2,7),(2,2),(4,2),(5,5),(4,3),(5,4),(3,4),(1,5),(6,1),(5,1),(3,5),(6,1).$$

The first doubles happened on the second roll, and there where two doubles altogether.

(b) What is the expected number of doubles in twelve rolls?

SOLUTION: We can think of each roll as a Bernoulli trial with success being a roll of doubles. The probability of success, i.e., of rolling doubles, is $p = 6/36 = 1/6$. Let Y be the binomial random variable whose value is the number of doubles in $n = 12$ rolls. The expected value of Y is

$$E(Y) = np = 12 \cdot \frac{1}{6} = 2.$$

(c) How long should it take to roll doubles?

SOLUTION: Let Z be the geometric random variable whose value is the number of times it takes to first roll doubles. Then

$$E[Z] = \frac{1}{p} = 6.$$

Problem 2.

(a) An airline has sold 205 tickets for a flight that can hold 200 passengers. Each ticketed person, independently, has a 5% chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

SOLUTION: We model this with a Bernoulli random variable with $p = 1 - 0.05 = 0.95$ and $n = 205$. The probability of more than 200 people showing up for a flight is, thus,

$$\sum_{k=201}^{205} \binom{205}{k} (0.95)^k (0.05)^{n-k} \approx 0.02236.$$

(b) If the same airline consistently oversells the flight from part (a) at the same rate, how many flights until we expect more ticketed passengers to show up than there are seats.

SOLUTION: Since the probability of overbooking is $p' := 0.02236$, we use a geometric random variable, and find the expected value to be

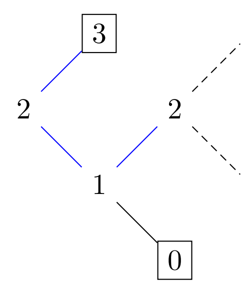
$$\frac{1}{p'} \approx 44.7.$$

PROBLEM 3.

Start with \$2. Flip a fair coin. If it comes up head, you win a dollar, and if it comes up tails, you lose a dollar. Stop when you either have no money left or you reach \$3.

(a) What is the probability of reaching \$3?

SOLUTION: Consider the probability tree shown below in which the the probability of following any edge is 1/2:



Let p be the probability of reaching \$3 starting at \$2. Following the blue edges on the tree, we see

$$p = \frac{1}{2} + \frac{1}{4}p.$$

Solving for p, we get

$$p = \frac{2}{3}.$$

(b) What is the expected total number of flips?

SOLUTION: Again refer to the probability tree. There is a 1/2 probability we stop after 1 flip, reaching \$3. There is a 1/4 probability that we stop after two flips, reaching \$0. Finally, there is a 1/4 probability that we take two flips, and start over again from \$2. Therefore, letting ℓ be the expected number of flips, we get

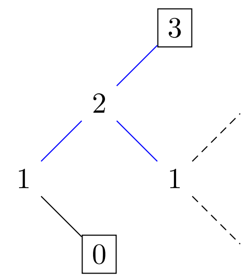
$$\ell = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + (2 + \ell)\frac{1}{4}.$$

Solving for ℓ, we get

$$\ell = 2.$$

(c) What is the answer to these questions if you start with \$1, instead?

SOLUTION: Now the probability tree is



Arguing as above, we have

$$p = \frac{1}{4} + \frac{1}{4}p \quad \Rightarrow \quad p = \frac{1}{3},$$

and

$$\ell = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + (2 + \ell)\frac{1}{4} \quad \Rightarrow \ell = 2.$$

Note there is a kind of symmetry between this case and the previous once that could have been exploited to answer this case without repeating the calculations.

One could generalize this problem: start with I dollars, flip a coin to add or subtract a dollar, stop if you reach D dollars or \$0. This is called the *gambler's ruin* problem. In general, the probability of success is $p = I/D$, and the expected length is $I(D - I)$.