Math 113, Monday Week 8

March 16, 2020

Dyck Paths

A Dyck path of length $2n$ is a monotonic (east/north=right/up) lattice path from $(0,0)$ to (n,n) that stays below the diagonal.

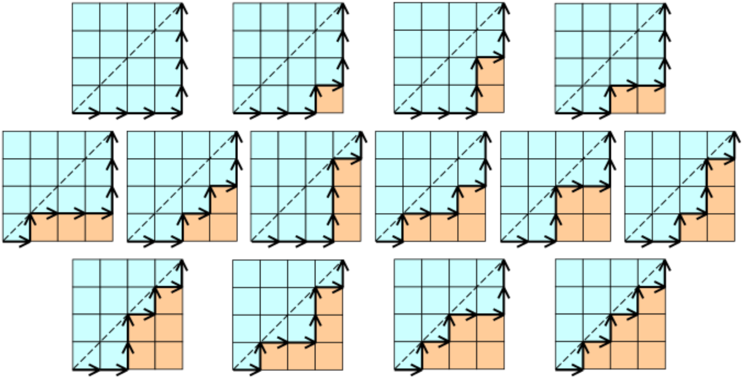


Figure 1. The 14 Dyck paths of length $2n$ where $n = 4$.

Theorem

The number of Dyck paths of length $2n$ is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

There are $\binom{2n}{n}$ monotonic lattice paths from $(0,0)$ to (n,n) in total.

To prove the theorem, we will partition the set of monotonic lattice paths into $n+1$ sets of equal size:

$$E_0 \amalg E_1 \amalg \cdots \amalg E_{n+1}$$

where E_0 is the set of Dyck paths.

Since each E_i has the same size, the result follows:

$$\binom{2n}{n} = |E_0|+|E_1|+\cdots+|E_{n+1}| = (n+1)|E_0| \Rightarrow |E_0| = \frac{1}{n+1}\binom{2n}{n}.$$

Define the exceedance of a monotonic lattice path to be the number of vertical steps of the path that are above the diagonal.

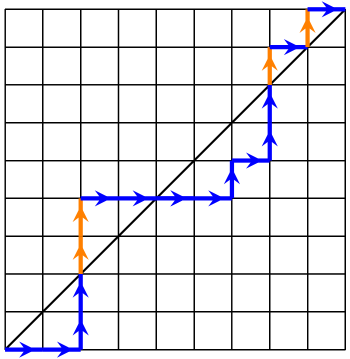


Figure 2. Monotonic lattice path with exceedance 4.

Define E_i to be the number of monotonic lattice paths from $(0, 0)$ to (n, n) with i exceedances.

As claimed: the E_i partition the full set of monotonic lattice paths, and E_0 is the set of Dyck paths.

Our goal is to define bijections

$$E_i \to E_{i+1}$$

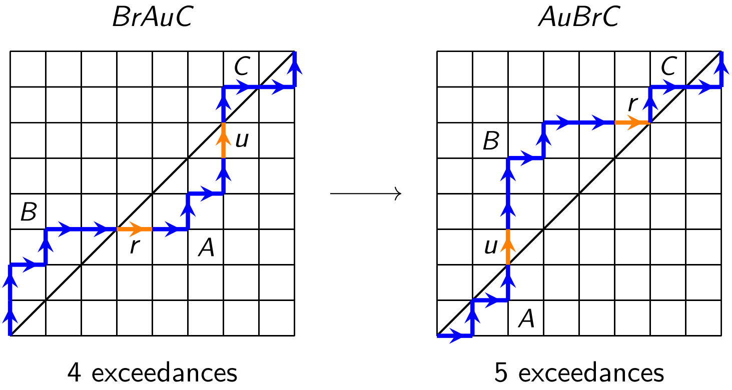
for $i = 0, \ldots, n$. This suffices to prove the theorem.

Each element of E_i can be written as $BrAuC$ where

$$
\begin{aligned}
r &= \text{first right step below the diagonal}\\
B &= \text{the part of the path, possibly empty, preceding } r\\
u &= \text{first up step after } r \text{ that touches the diagonal}\\
A &= \text{path between } r \text{ and } u\text{, again possibly empty}\\
C &= \text{the rest of the path.}
\end{aligned}
$$

Define $E_i \to E_{i+1}$ by

$$BrAuC \mapsto AuBrC.$$


BrAuC
B
r
A
u
C
4 exceedances
AuBrC
A
u
B
r
C
5 exceedances