E45 Find the number of rearrangements of the string 123456 in which none of the sequences 123, 321, 456, and 654 occur.

E46 Find the number of rearrangements of the string 1234567 that contain at least one of the sequences 123, 234, and 4567.

E47 Find the number of subsets of the set $\{1, 2, 3, 4, 5\}$ that do not contain any of the sets $\{1, 2, 3\}$, $\{2, 3, 4\}$, and $\{3, 4, 5\}$.

E48 Find the number of rearrangements of the string 111222333 that contain no three consecutive equal digits.

E49 Suppose you have a set of nine colored balls, including three red, two blue, two green, one white, and one yellow.

(a) How many ways can you select four?

(b) How many ways can you select five?

E50 Find the number of seven-digit combinations from the set $\{1, 2, 3, 4, 5\}$ if each digit can be selected at most twice. (*Hint:* Look for the easiest way to do this.)

E51 Repeat problem E50, assuming this time that each letter can be selected at most three times.

E52 Find the number of integers in the set $\{1, 2, 3, \ldots, 210\}$ that are divisible

(a) by exactly one of 2, 3, 5, and 7;

(b) by exactly two of 2, 3, 5, and 7.

F

Recurrence Relations

In this section, we will work with infinite sequences $(a_1, a_2, a_3, \ldots)$, usually with integer terms a_n. Sometimes it will be more convenient to have the numbering of the terms start at 0: $(a_0, a_1, a_2, \ldots)$.

A *recurrence relation* is a formula or rule by which each term of a sequence (beyond a certain point) can be determined using one or more of the earlier terms.

Examples (a) $7, 17, 27, 37 \ldots$

(b) $1, 10, 100, 1000 \ldots$

(c) $1, 3, 6, 10, 15 \ldots$

(d) $1, 2, 6, 24, 120 \ldots$

(e) $1, 1, 2, 3, 5, 8, 13 \ldots$

(f) $1, 1, 4, 10, 28, 76 \ldots$

(g) $1, 1, 1, 3, 5, 9, 17, 31 \ldots$

(h) $0, 1, 2, 9, 44, 265 \ldots$

F1 For each sequence (a)–(h), find a recurrence relation that describes the obvious (?) pattern. At what point in the sequence does the recurrence relation start to apply? Consider the first term of the sequence to be a_1.

F2 For sequences (a)–(d), find a nonrecursive formula for a_n. This means that a_n is expressed entirely in terms of n by a formula that does not depend on any of the other terms of the sequence.

Sequence (e) is known as the *Fibonacci sequence*. The usual notation for this sequence is

$$F_0 = 1, F_1, = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$$

and the recurrence relation for the Fibonacci sequence is

$$F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

Sequence (h) consists of the derangement numbers D_n that appeared in sections A and E:

$$D_1 = 0, D_2 = 1, D_3 = 2, D_4 = 9, \ldots$$

Later in this section we will see that the derangement numbers satisfy two recurrence relations, as follows.

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \text{for all } n \geq 3$$
$$D_n = nD_{n-1} + (-1)^n \quad \text{for all } n \geq 2.$$

We now look at a variety of combinatorial problems that can be solved using recurrence relations.

The Stamp Problem

Suppose we have 1¢, 2¢, and 5¢ stamps. The problem is to find the number of ways these can be arranged in a row so that they add up to a given value, n¢. The order of the stamps is taken into account. (Otherwise we might have used coins. That problem will be addressed in section G.) So, for example, $1 + 1 + 2$ is different from $1 + 2 + 1$. Let a_n represent the number of ways the stamps can add up to n¢. We assume that there is an unlimited supply of each type of stamp.

F3* Find a_1, a_2, a_3, and a_4 by counting directly.

Now, we introduce a recurrence relation for the calculation of a_n in general. We consider cases according to the value of the first stamp. If this value is 1, then the total value of the remaining stamps must be $n - 1$. Therefore the number of ways in which these remaining stamps can be selected is a_{n-1}. In

other words, there are a_{n-1} arrangements of stamps that have total value n and begin with a 1¢ stamp. Similarly, the number of arrangements that have total value n and begin with a 2¢ stamp is a_{n-2}. We assume that $n \geq 3$, so that the numbers a_{n-1} and a_{n-2} have already been determined. Finally, assuming that $n \geq 6$, there are a_{n-5} arrangements of stamps that have total value n and begin with a 5¢ stamp. From this analysis we arrive at the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + a_{n-5} \quad \text{for all } n \geq 6$$

F4 (a) Explain why $a_5 = a_4 + a_3 + 1$ and use this relation to find a_5.

(b) Use the recurrence relation to generate the values of a_n up to and including a_{10}.

F5* Use a recurrence relation to find the number of ways 1¢, 2¢, and 3¢ stamps can add up to 8¢.

Words with Limits on Consecutive Repetitions

In section A, we saw how we could easily count the words of a given length that use letters from a given set, in which no two consecutive letters are the same. All that was needed for that problem was the product rule. Then, in section B, we saw how to count rearrangements of a given string in which one particular symbol is not allowed to appear twice in a row. (See Standard Problem #4, in which the restriction is on the digit 1.)

Now, we will see how more complicated restrictions on consecutive repetitions can be handled using recurrence relations. To begin, consider the following problem.

Find the number of n-letter words using letters from the set $\{A, B\}$ in which there is a limit of 1 on consecutive As.

In other words, no two consecutive As can appear in the word.

F6* How does the preceding problem differ from Standard Problem #4, aside from the obvious change from digits to letters?

Notation

w_n = the number of n-letter words satisfying the given condition

a_n = the number of words counted by w_n that start with A

b_n = the number of words counted by w_n that start with B

Obviously, $a_n + b_n = w_n$.

F7 By directly counting words, fill in the values for all $n \leq 4$ in the following table.

n	1	2	3	4	5	6	7	8
a_n								
b_n								
w_n								

F8 (a) Explain why $a_n = b_{n-1}$ for all $n \geq 2$.
(b) Explain why $b_n = w_{n-1}$ for all $n \geq 2$.
(c) Use these relations to fill in the rest of the table in problem F7.

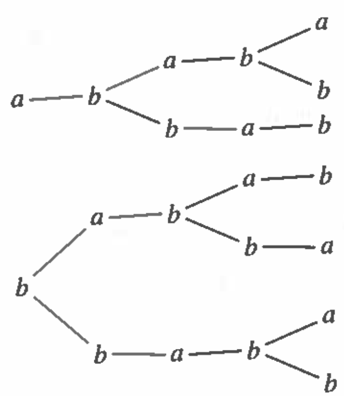
F9 Use the relations established in problem F8 to show that $w_n = w_{n-1} + w_{n-2}$ for all $n \geq 3$.

F10 What are a_n, b_n and w_n, as terms in the Fibonacci sequence?

F11* Find the number of subsets A of the set of digits $\{0, 1, 2, \ldots, 9\}$ such that A contains no two consecutive digits.

Next, we consider a variation on this problem, placing a limit of 2 on consecutive Bs while keeping the limit of 1 on consecutive As. As before, w_n counts the allowable words of length n, while a_n and b_n count the allowable words starting with A and B respectively.

F12 (a) Explain how the given conditions on consecutive equal letters are reflected in the following two tree diagrams.



(b) Use the tree diagrams to fill in the following table.

n	1	2	3	4	5
a_n					
b_n					
w_n					

F13* (a) Explain why

$$a_n = b_{n-1} \qquad \text{for all } n \geq 2$$

and

$$b_n = a_{n-1} + a_{n-2} \quad \text{for all } n \geq 3.$$

(*Suggestion:* Think in terms of tree diagrams.)

(b) Show that

$$a_n = a_{n-2} + a_{n-3}$$

$$b_n = b_{n-2} + b_{n-3}$$

and

$$w_n = w_{n-2} + w_{n-3} \quad \text{for all } n \geq 4.$$

(c) Find w_{12}.

(*Note:* The two equations in (a) above form a *system of recurrence relations.* Although it is easy enough to derive the recurrence relations in (b), we could just as well use the system in (a) to generate further values of a_n and b_n.)

Finally, we look at one more variation on this problem. Keeping the limit of 1 on consecutive As and 2 on consecutive Bs, we include three more letters C, D, and E with no limits on consecutive occurrences. Using the notation c_n, d_n, and e_n, notice that $c_n = d_n = e_n$ for all $n \geq 1$. Also, we introduce the notation a'_n, b'_n, etc., to represent the number of words of a given length that do *not* begin with a particular letter. For example,

$$a'_n = w_n - a_n \quad \text{and} \quad a'_{n-1} = w_{n-1} - a_{n-1}.$$

F14 Show that

$$a_n = a'_{n-1} \qquad \text{for all } n \geq 2$$

$$b_n = b'_{n-1} + b'_{n-2} \quad \text{for all } n \geq 3$$

and

$$c_n = w_{n-1} \qquad \text{for all } n \geq 2$$

F15* Fill in the following table, using the system of recurrence relations in problem F14. Remember the *D*s and *E*s when finding w_n.

n	1	2	3	4	5
a_n					
b_n					
c_n					
w_n					

Solving a Recurrence Relation

In certain cases, there is a method for finding an explicit formula for the nth term a_n in a sequence satisfying a recurrence relation. This method applies when the recurrence relation has the form

$$a_n = c_1a_{n-1} + c_2a_{n-2}$$

where c_1 and c_2 are constants (that is, they do not depend on n). The result is a formula for a_n that does not refer to earlier terms in the sequence. So, for example, the method applies to the Fibonacci sequence (where $c_1 = c_2 = 1$), but not to the derangement numbers (since $c_1 = c_2 = n - 1$ in that case).

The method consists of two steps, as indicated in the following algorithm.

Algorithm for solving a recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2}$$

given starting values a_0 and a_1:

Step 1
Find all values of r such that the geometric sequence $(1, r, r^2, \ldots, r^n, \ldots)$ satisfies the same recurrence relation.

Step 2
If two distinct r values α and β are obtained in step 1, set

$$a_n = \alpha^nA + \beta^nB;$$

If only one value $r = \alpha$ is obtained, set

$$a_n = \alpha^n(A + nB)$$

In either case, find constants A and B by using the starting values a_0 and a_1.

To be more specific, Step 1 amounts to solving the equation $r^2 = c_1r + c_2$ for r, while in Step 2 we solve for A and B in one of the systems of equations

$$\boxed{\begin{aligned} A + B &= a_0 \\ \alpha A + \beta B &= a_1 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} A &= a_0 \\ \alpha(A + B) &= a_1 \end{aligned}}$$

As an example, consider the sequence $1, 1, 3, 5, 11, \ldots$, in which $a_0 = a_1 = 1$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$.

F16 Use the algorithm to find a general formula for a_n. Check that this gives the correct result for a_7.

F17 (a) Find α and β in the case of the Fibonacci sequence. Let α represent the larger root.

(b) Without finding A and B, explain why α^nA is a good approximation to F_n for large values of n.

(c) What can you say about the ratio F_{n+1}/F_n as $n \to \infty$?

F18* Find A and B such that $F_n = \alpha^nA + \beta^nB$. Use a calculator to check that this gives the correct result for F_{10}. What is F_{25}?

F19 Find a formula for a_n satisfying $a_0 = 1, a_1 = 0$, and $a_n = 4(a_{n-1} - a_{n-2})$ for $n \geq 2$.

Recurrence relations of the form $a_n = c_1a_{n-1} + c_2a_{n-2}$, for fixed constants c_1 and c_2, are referred to as *linear with constant coefficients*. More generally, this term applies to recurrence relations of the form $a_n = c_1a_{n-1} + \cdots + c_ma_{n-m}$, for any fixed m and constants $c_1 \ldots c_m$. The algorithm we used for problems F16–19 can be adapted to solve any linear recurrence relation with constant coefficients. Some examples in which $m = 3$ appear in problems F40 and F41.

Derangement Numbers

Earlier in this section, we observed the following two recurrence relations satisfied by the derangement numbers.

(1) $D_n = (n-1)(D_{n-1} + D_{n-2})$ for all $n \geq 3$.

(2) $D_n = nD_{n-1} + (-1)^n$ for all $n \geq 2$.

Now, we will see why these are true, explaining the first combinatorially and then deriving the second from the first.

Looking at the case $n = 5$ to illustrate the idea, consider a derangement of the string 12345. The first digit can be 2, 3, 4, or 5. Suppose it is 2, and let $wxyz$ represent the last four digits. There are two cases:

If $w = 1$, then xyz is a derangement of 345; therefore, xyz can be chosen in D_3 ways;

If $w \neq 1$, then $wxyz$ is a derangement of 1345; therefore, $wxyz$ can be chosen in D_4 ways.

We conclude that the total number of derangements of 12345 that begin with 2 is $D_3 + D_4$.

F20 Show that the same number of derangements result from each of the other possible choices for the first digit in the derangement of 12345.

The ultimate result is that $D_5 = 4(D_3 + D_4) = 4(2 + 9) = 44$.

F21* Show by a similar argument that, in general,

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \quad \text{for all } n \geq 3$$

This explains recurrence relation (1) for the derangement numbers. To establish relation (2), set

$$a_n = D_n - nD_{n-1} \quad \text{for all } n \geq 2$$

Then, we must show that $a_n = (-1)^n$.

F22 (a) Check that $a_2 = 1$.

(b) Use recurrence relation (1) to show that $a_n = -a_{n-1}$ for all $n \geq 3$.

It follows that $a_n = (-1)^n$, and, therefore, the derangement numbers satisfy recurrence relation (2).

F23 (a) A standard deck of 52 distinct cards is shuffled. Which of the following is more likely to happen? No card is in its original position; or exactly one card is in its original position.

(b) Answer the same question in (a) if one joker is added to the deck.

Section Summary

A *recurrence relation* is a formula or rule by which each term of a sequence can be determined using one or more of the earlier terms. Examples are $F_n = F_{n-1} + F_{n-2}$ for the Fibonacci sequence and $D_n = (n-1)(D_{n-1} + D_{n-2})$ or $D_n = nD_{n-1} + (-1)^n$ for the sequence of derangement numbers. Other applications include counting the ways stamps having specified values can add up to a given total value, and counting words with limits on consecutive repetitions of particular letters.

Recurrence relations of the form $a_n = c_1a_{n-1} + c_2a_{n-2}$, for fixed constants c_1 and c_2, are *linear with constant coefficients* and can be solved algebraically, resulting in an explicit formula for a_n as a function of n. More general linear recurrence relations with constant coefficients $a_n = c_1a_{n-1} + \cdots + c_ma_{n-m}$ are solvable by a generalization of the same method.

More Problems

F24 Find the number of ways a row of stamps can be worth a total of 15 cents, using stamps worth 2, 3, 5, and 7 cents each.

F25 (a) Find the number of ways a row of stamps can be worth a total of n cents, for each $n \leq 8$, if all positive integer values are allowed for the individual stamps.

(b) From the pattern observed in (a), guess what is true for all n. Can you prove it?

F26 Repeat problem F25 with the values of the individual stamps restricted to odd numbers.

F27 Let w_n count the n-letter words that use letters from $\{A, B, C, D\}$, with a limit of 1 on consecutive As and a limit of 1 on consecutive Bs. Use an appropriate system of recurrence relations to fill in the following table, in which a_n and c_n represent the number of allowable words starting with A and C respectively.

n	1	2	3	4	5
a_n					
c_n					
w_n					

F28 For each $n \le 6$, find the number of n-letter words that use letters from $\{A, B, C\}$, if there is a limit of 1 on consecutive As and on consecutive B's, and a limit of 2 on consecutive Cs. (*Suggestion:* If necessary, look back at problems F12 through F14.)

F29 Repeat problem F28 with a limit of 3 on consecutive As and no limit on consecutive Bs and Cs.

F30 Let w_n count the n-letter words that use letters from $\{A, B, C, D, E\}$ with a limit of 2 on consecutive equal letters, and let a_n count the allowable words starting with A.

(a) Show that a_n and w_n satisfy the recurrence relations

$$a_n = 4(a_{n-1} + a_{n-2})$$

$$w_n = 4(w_{n-1} + w_{n-2}) \quad \text{for all } n \ge 3$$

(b) Find w_7. Use a calculator.

F31 Let w_n count the n-letter words that use letters from $\{A, B\}$ with a limit of k on consecutive equal letters, for some fixed positive integer k.

(a) Find a recurrence relation satisfied by w_n.

(b) Find w_{10} if $k = 3$.

F32 Generalize the result in problem F31(a), finding a recurrence relation for the number of n-letter words that use letters from an m-letter set, with a limit of k on consecutive equal letters.

F33 (a) Show that if a sequence satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, then it also satisfies $a_n = 2a_{n-1} - a_{n-4}$.

(b) Find a simpler recurrence relation for w_n in problem F32.

F34 Find the number of subsets A of the set of digits $\{0, 1, \ldots, 9\}$ having the property that among every four consecutive digits, A contains at least one digit and at most three.

F35 A subset of $\{1, 2, 3, \ldots, n\}$ is selected at random. If the set includes any three consecutive numbers, you win. How large must n be so that your probability of winning is greater than $\frac{1}{2}$?

F36 Find a formula for a_n satisfying $a_0 = 1, a_1 = 2$, and $a_n = a_{n-1} + 6a_{n-2}$, for all $n \ge 2$.

F37 (a) Find a formula for a_n satisfying $a_0 = 1, a_1 = 3$, and $a_n = 2a_{n-1} + a_{n-2}$, for all $n \ge 2$.

(b) Show that a_n is the nearest integer to $\alpha^n A$ for all $n \ge 0$, where α is the larger r value found in (a).

F38 Find a formula for a_n satisfying $a_0 = a_1 = 1$, and $a_n = -(2a_{n-1} + a_{n-2})$, for all $n \ge 2$.

F39 Find a formula for a_n satisfying $a_0 = a_1 = 1$, and $a_n = 2(a_{n-1} + a_{n-2})$, for all $n \ge 2$.

For problems F40 and F41, guess how to extend the algorithm for solving recurrence relations to apply to a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$.

F40 Find a formula for a_n satisfying $a_0 = 4, a_1 = -4, a_2 = 0$, and $a_n = 7a_{n-2} - 6a_{n-3}$, for all $n \ge 3$.

F41 Find a formula for a_n satisfying $a_0 = 0, a_1 = -1, a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} - a_{n-3}$ for all $n \ge 3$.

F42 (a) Let $a_0 = 2, a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \ge 2$. Find constants A and B such that $a_n = AF_n + BF_{n+1}$ for all $n \ge 0$, where F_n and F_{n+1} are terms of the Fibonacci sequence with the usual notation.

(b) Show that any sequence satisfying the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for all $n \ge 2$ can be expressed in the form $a_n = AF_n + BF_{n+1}$.

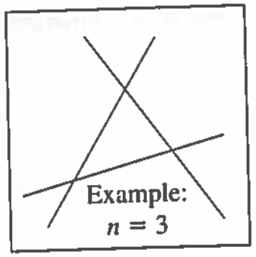
F43 Use the result of problem F10 and Standard Problem #4 to show that

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots \qquad \text{for all } n \ge 0$$

Counting Regions

Suppose we have n lines in a plane in general position, which means that none are parallel to each other and that no three of these lines intersect at single point.

The problem is to find the number of regions that these lines divide the plane into. Call this number r_n. Then $r_1 = 2, r_2 = 4, r_3 = 7$.



F44 Find a recurrence relation for r_n. (*Hint:* How many new regions are created by the nth line?) Use it to find r_{10}.

F45 Find a nonrecursive formula for r_n. (*Hint:* see what happens when you subtract 1 from each term.)

F46 Suppose there are n circles in a plane in general position, which means that any two of the circles intersect in exactly two points, and no three circles intersect at a single point.

(a) Find a recurrence relation for r_n, the number of regions formed by these circles.

(b) Try to find a nonrecursive formula for r_n.

F47 In the algorithm for solving a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2}$, suppose that the equation $r^2 = c_1r + c_2$ has two real solutions α and β or one solution α. Show that if $c_1 \neq 0$, then a_{n+1}/a_n approaches a finite limit as $n \to \infty$. What is this limit? (*Suggestion:* First show that if there are two solutions, then α and β cannot have the same absolute value, and therefore either α/β or β/α has absolute value < 1. In any case, use the result of the algorithm.)

F48 Show that the sequence that satisfies

$$a_1 = a_2 = 1 \quad \text{and} \quad a_n = (n+1)a_{n-1} - (n-1)a_{n-2} \quad \text{for all } n \geq 3$$

must also satisfy

$$a_n = na_{n-1} - 1 \quad \text{for all } n \geq 2.$$

(*Hint:* If necessary, see problem F22.)

F49 Let w_n represent the number of n-letter words that use letters from the set $\{A, B, C\}$ and contain an odd number of As. Let a_n be the number of these words that start with A.

(a) Show that

$$a_n = 3^{n-1} - w_{n-1}$$

and

$$w_n = a_n + 2w_{n-1}$$

for all $n \geq 2$.

(b) Find a recurrence relation satisfied by w_n, for all $n \geq 2$.

(c) Find a nonrecursive formula for w_n, for all $n \geq 1$.

F50 Let w_n represent the number of n-letter words that use letters from the set $\{A, B, C, D, E\}$ and contain an even number of As (possibly 0).

(a) Find a recurrence relation satisfied by w_n, for all $n \geq 2$.

(b) Calculate w_n, for all $n \leq 6$.

F51 Let S be a set of $2n$ elements and let p_n represent the number of partitions of S into n parts, with two elements in each part. Then $p_1 = 1$.

(a) Explain why $p_n = (2n-1)p_{n-1}$, for all $n \geq 2$.

(b) Find p_5. Check your answer by using Standard Problem #15.

F52 Let S be a set of $3n$ elements and let p_n represent the number of partitions of S into n parts, with three elements in each part.

(a) Find a recurrence relation for p_n.

(b) Find p_4. Check your answer by using Standard Problem #15.