MATH 113: DISCRETE STRUCTURES
FINAL EXAM REVIEW

The final exam is cumulative, but will have a slight emphasis on number theory topics covered after the second exam. The final is two hours long and will take place Monday, December 11, 1–3P.M. in Physics 123. You may bring one two-sided sheet of notes (US letter size or A4 paper) to the exam. No other resources (including books, other notes, electronic aids, or consultation) are permitted. Your dictionary for the term is due at the start of the exam.

The following problems are for practice and are intended to help you review for the exam. They are not to be turned in and will not be graded, though I am happy to provide feedback if you have questions about a solution.

At least one of these problems will appear on the exam, but most of these problems are *harder* than exam problems.

Problem 1. What is the number of 20-digit integers in which no two consecutive digits are the same?

Problem 2. Prove that
$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$
Represent this identity diagrammatically in Pascal's triangle.

Problem 3. Prove that $1 + 3 + 9 + 27 + \cdots + 3^{n-1} = (3^n - 1)/2$.

Problem 4. For which m and n does
$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m} \text{ ?}$$
Try to find a combinatorial proof of your assertion.

Problem 5. How many permutations $\pi : \underline{2n} \to \underline{2n}$ only take even numbers to odd numbers and odd numbers to even numbers?

Problem 6. Suppose that a and b are positive integers.

(a) Determine the number of sequences with a 0's and b $1's$.
(b) Consider two such sequences equivalent if and only if you can arrive at one from the other by cyclic permuting the string. Suppose that $\gcd(a, b) = 1$ and determine the number of equivalence classes of such sequences, proving any assertions you make.
(c) Does your formula from (b) still work when $\gcd(a, b) \neq 1$? Check it with $a = b = 2$.

Problem 7. We select a subset A of the set $\underline{100}$ randomly and uniformly. What is the probability that

(a) A has an even number of elements?
(b) both 1 and 100 belong to A?
(c) the largest element of A is 50?
(d) A has at most 2 elements?

Problem 8. We flip a coin n times ($n \geq 1$). For which values of n are the following pairs of events indpendent?

(a) The first coin flip was heads; the number of all heads was even.
(b) The first coin flip was heads; the number of all heads was more than the number of tails.
(c) The number of heads was even; the number of heads was more than the number of tails.

Problem 9. Suppose that there are N possible birthdays distributed randomly and uniformly amongst a population of n individuals. (Perhaps $N = 365$, but let's stick with just N for the moment.)

(a) Suppose that Alma and Bodie are two of the n people. What is the probability that Alma and Bodie don't share a birthday?
(b) What is the probability that all $n-1$ of the other people don't share Alma's birthday?
(c) What is the expected number of people who do not share anyone else's birthday?
(d) What is the expected number of people who do share someone else's birthday?
(e) Let $N = 365$ and use a computer or calculator to determine the smallest n for which we expect 6 amongst n people to share birthdays.

Problem 10. A biased coin comes up heads with probability p, $0 \leq p \leq 1$, and tails with probability $q = 1 - p$. If the coin is flipped 5 times, what is the probability that it comes up heads an odd number of times?

Problem 11. Prove that every prime $p > 3$ gives a remainder of 1 or -1 when divided by 6.

Problem 12. Show that if $p > 2$ is prime, then $\overline{2}^{-1} = \overline{((p+1)/2)}$ in $\mathbb{Z}/p\mathbb{Z}$.

Problem 13. Recall that the Fibonacci numbers F_n satisfy the recurrence relation $F_{n+1} = F_n + F_{n-1}$ with initial values $F_0 = 0$, $F_1 = 1$. The Lucas numbers L_n satisfy the same recurrence ($L_{n+1} = L_n + L_{n-1}$) but have $L_0 = 2$, $L_1 = 1$.

(a) Prove that $\gcd(F_{3k}, L_{3k}) = 2$ for all integers $k \geq 0$.
(b) Prove that if $3 \nmid n$, then $\gcd(F_n, L_n) = 1$.
(c) Prove that $L_{6k} \equiv 2 \pmod 4$ for all integers $k \geq 0$.

Problem 14. Again for Fibonacci numbers F_n and Lucas numbers L_n, prove the following identities:

(a) $F_{2n} = F_n L_n$;
(b) $2L_{k+n} = 5F_k F_n + L_k L_n$;
(c) $L_{4k} = L_{2k}^2 - 2$.

Problem 15 (The Freshman's Dream). Suppose p is prime and that $a, b \in \mathbb{Z}$. Prove that

$$(a+b)^p \equiv a^p + b^p \pmod p.$$

(Many freshmen falsely believe that such identities hold over $\mathbb{Z}$ — they don't! — and thus this identity is often referred to as the Freshman's Dream.)

Problem 16. Fix a prime p. A polynomial in x with coefficients in $\mathbb{Z}/p\mathbb{Z}$ is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where each $a_i \in \mathbb{Z}/p\mathbb{Z}$. Such a polynomial is a *monic quadratic* if $n = 2$, and $a_2 = 1$, *i.e.*, if it is of the form $x^2 + a_1 x + a_0$ with $a_0, a_1 \in \mathbb{Z}/p\mathbb{Z}$. Such a polynomial is *reducible* if it can be factored, $x^2 + a_1 x + a_0 = (x - b_0)(x - b_1)$ with $b_0, b_1 \in \mathbb{Z}/p\mathbb{Z}$; otherwise, it is called *irreducible*.

(a) What is the total number monic quadratic polynomials in x with coefficients in $\mathbb{Z}/p\mathbb{Z}$?
(b) How many of these polynomials are irreducible? If a_0 and a_1 are chosen uniformly randomly, what is the probability of getting an irreducible monic quadratic polynomial?

Problem 17. In this problem, let ϕ denote Euler's phi function.

(a) Suppose that p, q are prime integers. Prove that $\phi(pq) = \phi(p)\phi(q)$.
(b) Prove that $\phi(mn) = d\phi(m)\phi(n)/\phi(d)$ where $m, n \in \mathbb{Z}$ and $\gcd(m, n) = d$.
(c) Prove that if $a \mid b$, then $\phi(a) \mid \phi(b)$.