MATH 113: DISCRETE STRUCTURES
EXAM 1 REVIEW

For the first exam, you are responsible for the content covered in class and our reading assignments through Monday, September 25. Topics include: sets, functions, the number of subsets, sequences, permutations, number of ordered subsets, number of subsets of a given size, binomial coefficients and basic relations between them, mathematical induction, the inclusion-exclusion principle, the pigeonhole principle, and the binomial theorem.

The exam is an in-class, 50-minute exam. You may bring one two-sided sheet of notes (US letter size or A4 paper). No other resources (including books, other notes, electronic aids, or consultation) are permitted.

The following problems are for practice and are intended to help you review for the exam. They are not to be turned in and will not be graded, though I am happy to provide feedback if you have questions about a solution.

One of these problems will appear on the exam, but most of these problems are *harder* than exam problems.

Problem 1 (*DM:EB* 1.8.27). Alice has 10 balls (all different). First, she splits them into two piles; then she picks one of the piles with at least two elements, and splits it into two; she repeats this until each pile has only one element.

(a) How many steps does this take?
(b) Show that the number of different ways in which she can carry out this procedure is

$$\binom{10}{2} \cdot \binom{9}{2} \cdots \binom{3}{2} \cdot \binom{2}{2}.$$

(*Hint*: Imagine the procedure backward.)

Problem 2 (*DM:EB* 1.8.32). Find all positive integers a, b, and c for which

$$\binom{a}{b}\binom{b}{c} = 2\binom{a}{c}.$$

Problem 3 (*DM:EB* 1.8.34). Twenty people are sitting around a circular table. How many ways can we choose 3 people, no two of whom are neighbors?

Problem 4 (*DM:EB* 2.5.3). Prove the identity

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$$

for $n \geq 1$.

Problem 5 (*DM:EB* 2.5.7). We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

Problem 6 (*DM:EB* 2.5.8). A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red, and 4 pairs of green socks.

(a) How many single socks do we have to take out to make sure that we take out two socks of the same color?
(b) How many single socks do we have to take out to make sure that we take out two socks with different colors?

Problem 7. How many nonempty subsets of $\underline{10}$ have the product of their elements even?

Problem 8. A *Shidoku board* is a 4×4 grid of numbers in which each of the numbers $1, 2, 3, 4$ appears exactly once in each row, column, and in each the northwest, northeast, southwest, and southeast 2×2 sub-grids. For example,

$$\begin{pmatrix} 4 & 3 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ 1 & 4 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

are Shidoku boards. How many different Shidoku boards are there?

Problem 9. Suppose that A and B are finite sets with $|A| = |B|$ and that $f : A \to B$ is a function. Prove that f is injective if and only if f is surjective.

Problem 10. A grade school class has three sports teams. For any two students in the class, there is at least one team such that the two students are members of that team. Prove that there is a team that contains at least two-thirds of the students of the class.

Problem 11. Let n be odd and suppose that

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

is a permutation of $\underline{n}$. Prove that the product $(x_1 - 1)(x_2 - 2) \cdots (x_n - n)$ is even. Is the result necessarily true if n is even? Give a proof or counterexample.

Problem 12 (*DM:EB* 3.8.12). Use the binomial theorem to prove that

$$1 + \binom{n}{1}2 + \binom{n}{2}4 + \cdots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n = 3^n.$$

Try to find a combinatorial proof as well.