MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 4 SUPPLEMENT

Let's first show that the identity

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}$$

holds for integers $n \geq 2$.

Proof 1 (double counting). Observe that the right-hand side gives the number of 3-subsets of $\underline{n+2} = \{1, 2, \ldots, n+2\}$. We show that the sum on the left-hand side counts the same objects.

Given a 3-subset A of $\underline{n+2}$, write it as $\{a, b, c\}$ where $a < b < c$ by ordering its elements. For a given integer b between 2 and $n+1$, we count the number of 3-subsets with b as their middle element. The first element, a, can be any of the $b-1$ elements of $\{1, 2, \ldots, b-1\}$, while the final element, c, can be any of the $n+2-b$ elements of $\{b+1, b+2, \ldots, n+2\}$. (Stop and think for a while if it is not immediately clear that there are $n+2-b$ such elements.) Thus there are $(b-1)(n+2-b)$ 3-subsets of $\underline{n+2}$ with b as middle element. Letting b range from 2 (the smallest possible middle element) to $n+1$ (the largest possible middle element), we count a total of

$$1 \cdot n + 2 \cdot (n-1) + \cdots + (n-1) \cdot 2 + n \cdot 1$$

3-subsets of $\underline{n+2}$. (Note that we have not double-counted anything because subsets with distinct middle elements are distinct.) We conclude that

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}$$

as desired. □

Proof 2 (induction). We first check the base case, $n = 2$. For this n, the left-hand side is $1 \cdot 2 + 2 \cdot 1 = 4$ and the right-hand side is $\binom{4}{3} = 4$, so the identity holds.

Now assume that for some $n \geq 2$ we have

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}.$$

Replacing n with $n+1$ on the left-hand side we get

$$1 \cdot (n+1) + 2 \cdot (n+1-1) + 3 \cdot (n+1-2) + \cdots + (n+1-1) \cdot 2 + (n+1) \cdot 1$$

Here each term is of the form $k \cdot (n+2-k)$ where k ranges from 1 to $n+1$. Note that we can rewrite $k \cdot (n+2-k)$ as $k \cdot (1 + (n+1-k)) = k + k \cdot (n+1-k)$. Adding these terms up (first the k summand, then the $k \cdot (n+1-k)$ summand), we get

$$(1 + 2 + \cdots + n + (n+1)) + (1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1).$$

By the induction hypothesis, the second term is just $\binom{n+2}{3}$. By the combinatorial argument we gave in class, the first term is $\binom{n+2}{2}$. We can thus conclude that

$$1 \cdot (n+1) + 2 \cdot (n+1-1) + 3 \cdot (n+1-2) + \cdots + (n+1-1) \cdot 2 + (n+1) \cdot 1 = \binom{n+2}{2} + \binom{n+2}{3}.$$

Finally, by Pascal's identity, the final sum is $\binom{n+3}{3} = \binom{(n+1)+2}{3}$, as desired. By mathematical induction, we now know that the identity holds for all $n \geq 2$. □

Let's now show that for $1 \le k \le n$,

$$n^k \le \binom{n}{k} k^k.$$

Before getting into the proof, note that if we divide both sides by k^k (a positive number), we get

$$\frac{n^k}{k^k} \le \binom{n}{k},$$

which is equivalent to

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k}$$

for $1 \le k \le n$, which, if nothing else, is an orthographically attractive inequality. (*Challenge*: Run some computer experiments to see how effective the bound $(n/k)^k$ is on $\binom{n}{k}$. Are they close or far apart as n gets large?)

Proof. Our strategy is as follows: given $1 \le k \le n$, we construct sets X and Y such that $|X| = \binom{n}{k} k^k$ and $|Y| = n^k$. We will then exhibit a surjection $f : X \to Y$. This guarantees the inequality $|X| \ge |Y|$, which is equivalent to $n^k \le \binom{n}{k} k^k$.

Fix integers k and n with $1 \le k \le n$. We take Y to be the collection of words in $\underline{n} = \{1, \ldots, n\}$ of length k. In other words,

$$Y = \{(a_1, a_2, \ldots, a_k) \mid a_1, \ldots, a_k \in \underline{n}\}.$$

Since there are n choices for each letter in each length k word, we have $|Y| = n^k$.

Now let X be the collection of pairs $(A, (a_1, \ldots, a_k))$ where A is a k-subset of $\underline{n}$ and $a_1, \ldots, a_k \in A$. There are $\binom{n}{k}$ ways to select the first term, A, and then k^k possible length k words $(a_1, \ldots, a_k)$ constructed from elements of A. Thus $|X| = \binom{n}{k} k^k$.

Finally, we construct a surjective function $f : X \to Y$ by declaring that

$$f((A, (a_1, \ldots, a_k))) = (a_1, \ldots, a_k).$$

This is well-defined for any $(A, (a_1, \ldots, a_k)) \in X$ as the word $(a_1, \ldots, a_k)$ is constructed from a subset of $\underline{n}$. Moreover, for each word $(a_1, \ldots, a_k) \in Y$, we can construct a pair $(A, (a_1, \ldots, a_k)) \in X$ such that $f((A, (a_1, \ldots, a_k))) = (a_1, \ldots, a_k)$ by choosing some A which contains $\{a_1, \ldots, a_k\}$. This means that f is surjective, so we have finished our proof! $\square$