MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 9 HANDOUT

Suppose events $A_1, A_2, \ldots, A_n$ partition our sample space S. This means that $A_1 \cup \cdots \cup A_n = S$ and $A_i \cap A_j = \varnothing$ for all $i \neq j$. Then the following statements are true for any event $C \subseteq S$ and any $1 \leq j \leq n$:

$$P(C) = P(C|A_1)P(A_1) + P(C|A_2)P(A_2) + \cdots + P(C|A_n)P(A_n),$$

$$P(A_j|C) = \frac{P(C|A_j)P(A_j)}{P(C|A_1)P(A_1) + P(C|A_2)P(A_2) + \cdots + P(C|A_n)P(A_n)}$$

Breaking with the terminology from the video lecture, let's call the first equality the *law of total probability* and the second one the *(generalized) Bayes' theorem*. Note that the numerator of the right-hand side of the second equation is, by definition, $(P(C \cap A_j)/P(A_j))P(A_j) = P(A_j \cap C)$, while the denominator is $P(C)$ by the law of total probability. Thus we have proven Bayes' theorem. Note that by rewriting the denominator as $P(C)$, we get the "standard" Bayes' theorem

$$P(A|C) = \frac{P(C|A)P(A)}{P(C)}$$

where $A \subseteq C$ is any event.

Problem 1. Should polygraph tests be admissible as evidence in courts? On a given question, let p be the event of a positive test (so the polygraph indicates that the subject is lying), and let n be the event of a negative test. Let T be the event that the subject is telling the truth, and let L be the event that the subject is lying. A study of polygraph reliability indicates that $P(p|L) = .88$ and $P(n|T) = .86$.

(a) Determine $P(n|L)$ and $P(p|T)$.
(b) Suppose that on a particular question, very few people have any reason to lie, so $P(T) = .99$ and $P(L) = .01$. If a subject produces a positive polygraph response on this question, what is the probability that they are telling the truth and the polygraph is incorrect?

The probability $P(T|p)$ is called the rate of *false positives*. Is this an acceptable rate of false positives?