MATH 113: DISCRETE STRUCTURES
FRIDAY WEEK 7 HANDOUT

Each of the following questions is meant to motivate a line of inquiry which might ultimately lead you to a conjecture in combinatorics. You might be able to prove the conjecture, and you should definitely try! You also might discover conjectures that have been stated for decades (or centuries) but haven't been proved yet. Maybe you'll conjecture (and prove) something totally new!

Question 1. Find more patterns in Pascal's triangle. State them precisely and try to prove them.

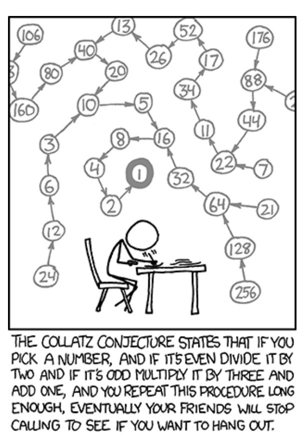
(a) Which binomial coefficients are even? What do they look like in Pascal's triangle?
(b) You've already found the sums of various lines of in Pascal's triangle and found answers like 2^n and the Fibonacci sequence. Add up other lines or line segments.
(c) Alternating sums along various lines? Products? Patterns in triangles of terms in Pascal's triangle?
(d) Can you find the Catalan numbers in Pascal's triangle? (See Question 4(e).)

Question 2. For each positive integer a, what is the number $N(a)$ of times that a appears in Pascal's triangle?

(a) If $a > 1$, then a appears in the a-th row of Pascal's triangle and does not appear anywhere below this row. Use this observation to get an upper bound on $N(a)$. What about lower bounds?
(b) Determine $N(a)$ for $1 \leq a \leq 16$. Any observations?
(c) Use a computer to extend your computation of $N(a)$ through a large range. Alternatively, look up OEIS-A003016. What do you think?

Question 3. Define a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n) = n/2$ when n is even and $f(n) = 3n+1$ when n is odd. Given a positive integer k define the sequence $(c_i(k))_{i=0}^{\infty}$ by $c_0(k) = k$ and $c_i(k) = f(c_{i-1}(k))$ for $i > 0$. This is called the *Collatz sequence* for k.

(a) Determine the behavior of $(c_i(1))$. Note that if $c_n(k) = 1$ then $c_{n+i}(k) = c_i(1)$.
(b) Determine the behavior of $(c_i(k))$ for $2 \leq k \leq 15$. What do all these sequences have in common?
(c) Make a conjecture and explore further.



Source: xkcd #710

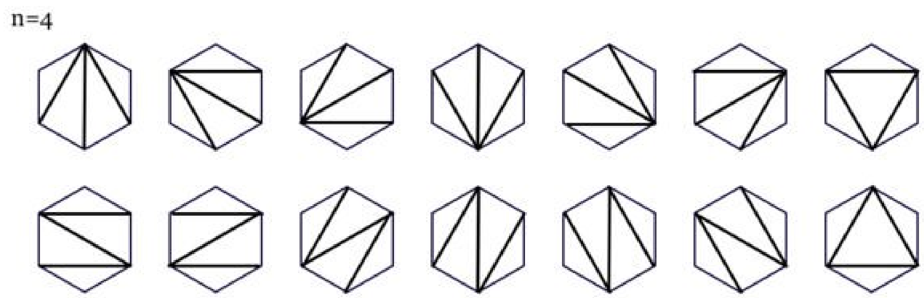
Question 4. The Catalan numbers are ubiquitous in combinatorics. They form a sequence $(b_n)_{n\geq 0}$ which counts all of the following things:

(i) The number of parenthesizations of a string of length $n+1$. For instance,

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d), \quad a(b(cd))$$

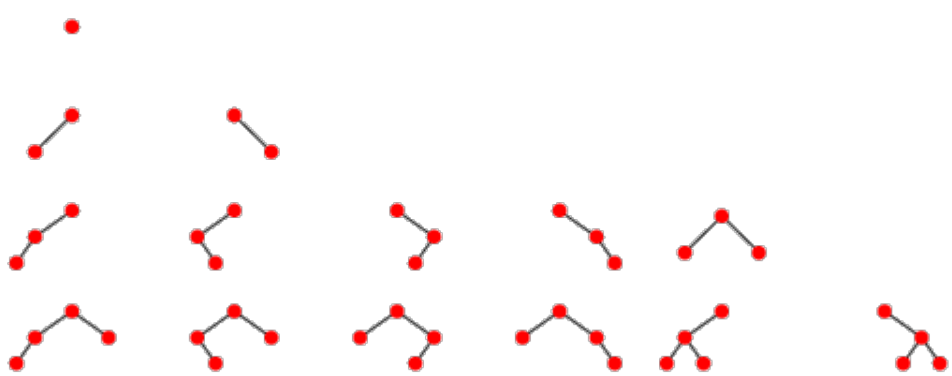
are all of the parenthesizations of a string of length 4, so $b_3 = 5$.

(ii) The number of triangulations of a regular $(n+2)$-gon by chords between vertices. For instance,



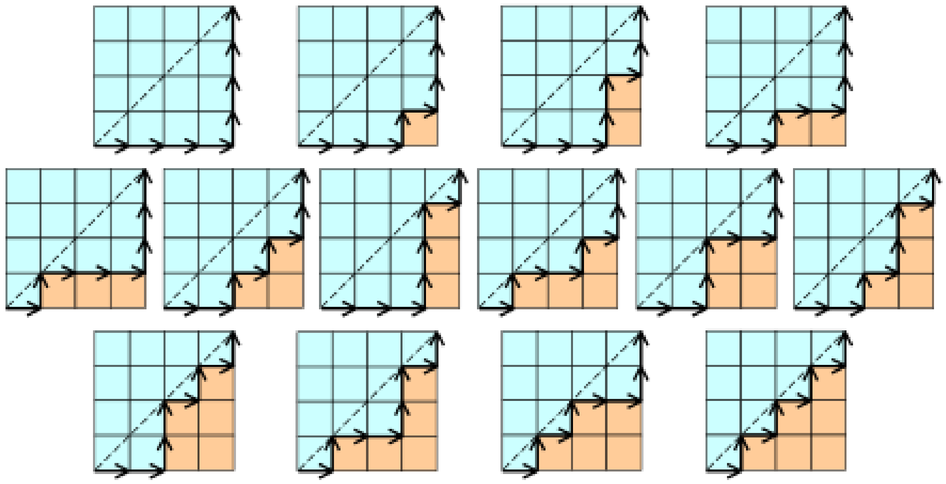
are all the triangulations of a hexagon, so $b_4 = 14$.

(iii) The number of binary trees with n vertices. A binary tree has the following recursive definition: a binary tree is either empty (and has 0 vertices), or consists of one distinguished vertex (the root) plus an ordered pair of binary trees (the left and right subtrees). We can draw these in the following fashion:



(iv) The number of non-crossing arc diagrams of size $2n$. These are systems of n non-intersecting semicircles in the upper half plane joining the points $(1,0)$, $(2,0)$, $\ldots$, $(2n,0)$.

(v) The number of lattice paths on an $n \times n$ grid from the bottom left corner to the top right corner that only go up and right and always stay below the diagonal. For instance,



also illustrates that $b_4 = 14$.

(a) Show that the above numbers really do coincide. There are two primary methods which you may find useful: (1) produce an explicit bijection, or (2) show that the objects being counted satisfy the recurrence relation $b_n = b_0 b_{n-1} + b_1 b_{n-2} + b_2 b_{n-3} + \cdots + b_{n-1} b_0$ for $n > 0$ and $b_0 = 0$.

(b) Consider a chord-triangulated $(n+2)$-gon. Mark one of the chords or edges in the triangulation and turn it into an arrow by designating one of its two endpoints the "head." How many such decorated chord-triangulated $(n+2)$-gons are there (in terms of b_n and n)?

(c) Consider a chord-triangulated $(n+3)$-gon with vertices labeled $1, 2, \ldots, n+3$ clockwise. Mark one of its edges (not chords) which is not the the edge with endpoints 1 and $n+3$. How many such decorated chord-triangulated $(n+3)$-gons are there (in terms of b_{n+1} and n)?

(d) Produce a bijection between the decorated diagrams of (b) and (c), thus resulting in a different recurrence relation between the Catalan numbers. Use this recurrence relation to prove that $b_n = \frac{1}{n+1}\binom{2n}{n}$.

(e) Can you find the Catalan numbers in Pascal's triangle? (Express b_n as a sum/difference of various binomial coefficients.)

(f) We can pair non-crossing arc diagrams of length $2n$ by flipping one upside down and gluing them together. This results in some number of closed loops. Which pairings result in a single loop? (This is would answer an open question about the so-called Temperley-Lieb monoid.)

(g) Make conjectures and explore further.