MATH 113: DISCRETE STRUCTURES
MONDAY WEEK 6 HANDOUT

In class, we provided a combinatorial proof for the identity

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}. \tag{1}$$

The idea was to count the number of $(k+1)$-element subsets of $\underline{n+1}$ in two ways. First, there is the obvious way, $\binom{n+1}{k+1}$, which is the the right-hand side of the identity. The less obvious way counts the number of such subsets with largest element L, where L ranges from $k+1$ to $n+1$. (Make sure you see why these are the smallest and largest possible largest elements of a $(k+1)$-subset of $\underline{n+1}$.) Now if L is in this range and is the largest element of a $(k+1)$-subset of $\underline{n+1}$, then all the other elements of the subset must be in $\underline{L-1}$, and there are k of them. Thus the number of $(k+1)$-subsets of $\underline{n+1}$ with largest element L is $\binom{L-1}{k}$. As L ranges though $k+1, k+2, k+3, \ldots, n+1$, we see these binomial coefficients take the values $\binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \ldots, \binom{n}{k}$. Adding these up, we find that the left-hand side of our identity counts the number of such subsets. (Make sure you also understand why we haven't overcounted anything!)

What follows is very Math 112-y, and not very Math 113-y. While the distinctions below are necessary for a formally verifiable inductive proof of (1), they are not very enlightening.

We can also prove this identity via mathematical induction, but let's first contemplate the unspoken hypotheses for our identity: for which natural numbers n and k does (1) hold? Our proof clearly works whenever $n \geq k$. Does the identity also hold when $n < k$? In this case, the right-hand side is 0 by convention. Meanwhile the right-hand side is an "empty sum": trying to increase indices from k to n when $n < k$ is impossible, so there are no terms to add up. For reasons that will become clear in Math 112, empty sums also evaluate to 0, so (1) holds for all natural numbers n and k. This is the statement we will prove via induction.

Proof by induction. For each natural number n, we consider the statement "equation (1) holds for all natural numbers k (and our fixed natural number n)." When $n=0$, we see that the right-hand side is 1 if $k=0$, and is 0 otherwise. If $k=0$, the left-hand side is $\binom{0}{0}=1$; if $k \neq 0$, the left-hand side is an empty sum which evaluates to 0. In both cases, (1) is true so our base case holds.

Now assume that for some natural number n, (1) is true for all k. It follows that for any k,

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}+\binom{n+1}{k}=\binom{n+1}{k+1}+\binom{n+1}{k}.$$

By Pascal's identity (*i.e.*, the Scissors Theorem), the right-hand side evaluates to $\binom{n+2}{k+1}$. Thus (1) holds when we replace n with $n+1$, and we have completed our inductive proof. $\square$

What happens if we try to use induction to prove that (1) holds for natural numbers $n \geq k$, the natural and interesting range of numbers for which to consider the identity? We can proceed in the same fashion, but we won't be able to apply the inductive hypothesis to the case $k=n+1$ (because then $k > n$). Observe, though, that replacing n and k with $n+1$ in (1) results in the purported identity $\binom{n+1}{n+1}=\binom{n+2}{n+2}$. This is in fact true because both left- and right-hand sides are equal to 1. Thus we can still push the proof through at the small cost of adding a special case to our argument.