MATH 113: DISCRETE STRUCTURES
FRIDAY WEEK 1 HANDOUT

Cartesian products. Given sets A, B we define $A \times B$, the *Cartesian product* of A and B, to be the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. In set constructor notation, this becomes

$$A \times B = \{(a, b) \ : \ a \in A, \ b \in B\}.$$

Problem 1. If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

Functions. A *function* $f : A \to B$ (with *domain* A and *codomain* B) is a subset $f \subseteq A \times B$ such that for every $a \in A$ there is precisely one term of the form $(a, b) \in f$; we write $b = f(a)$ or $f : a \mapsto b$ when $(a, b) \in f$.

Problem 2. Reconcile this definition of "function" with your preconceptions. How would you write the function $f(x) = x^2$ as a subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$?

Problem 3. Which of the following subsets of $\{1, 2, 3\} \times \{a, b, c, d\}$ are functions?
(a) $\{(1, a), (2, b), (3, d)\}$
(b) $\{(2, d), (3, c)\}$
(c) $\{(1, b), (2, c), (3, a), (2, d)\}$
(d) $\{(1, a), (2, a), (3, a)\}$

Problem 4. Functions $f : A \to B$ and $g : A \to B$ are equal if they are equal as subsets of $A \times B$. Check that this condition is equivalent to $f(a) = g(a)$ for all $a \in A$.

Problem 5. Suppose $|A| = m$, $|B| = n$. How many functions $A \to B$ are there?

Injections, surjections, and bijections. A function $f : A \to B$ is *injective* if $f(a) = f(a')$ implies $a = a'$. A function is *surjective* if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. A function is *bijective* if it is both injective and surjective.

Problem 6. Suppose $|A| = m$, $|B| = n$. How many injective functions $A \to B$ are there? (Make sure your answer makes sense when $m > n$.)

Schröder-Bernstein. The Schröder-Berstein theorem states that if there are injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$.

Problem 7. Prove the Schröder-Bernstein theorem in the special case in which both A and B are finite sets.[1]

[1] When A and B are infinite, the proof is more difficult and fairly subtle. If this sort of thing interests you, I encourage you to look up and understand a proof.