

# MATH 113: DISCRETE STRUCTURES

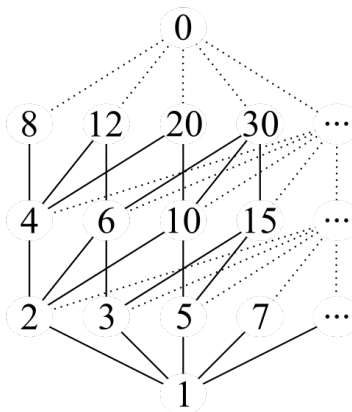
## MONDAY WEEK 11 HANDOUT

For integers  $a, b$ , we say that  $a$  divides  $b$  when an integer  $m$  exists such that  $b = am$ ; in this case we also say that  $b$  is a multiple of  $a$  and that  $a$  is a divisor of  $b$ .

*Question 1.* When does  $1 \mid b$ ?  $-1 \mid b$ ?  $a \mid 0$ ?  $a \mid a$ ?

*Problem 2.* Suppose that  $a \mid b$  and  $b \mid c$ . Prove that  $a \mid c$ .

This produces a *partial order* on  $\mathbb{N}$ , visualized in the following diagram.



*Question 3.* Where should you put 9 in the diagram?

*Problem 4.* Prove that if  $a \mid b$  and  $a \mid c$ , then  $a \mid b + c$  and  $a \mid b - c$ .

A natural number  $p > 1$  is *prime* if its only divisors are 1 and  $p$ . The fundamental theorem of arithmetic says that every positive integer is a product of primes, and that this factorization is unique up to reordering of the factors. For instance,  $6 = 2 \cdot 3$ ,  $1728 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^6 \cdot 3^3$  and  $825 = 3 \cdot 5 \cdot 5 \cdot 11 = 3 \cdot 5^2 \cdot 11$ . This probably seems like old hat, but not every number system has unique factorization! For instance,  $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$  supports addition and multiplication, but

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

Number theorists are quite interested in objects like  $\mathbb{Z}[\sqrt{-5}]$ , but we will limit our study to  $\mathbb{Z}$  where the fundamental theorem of arithmetic holds.

*Question 5.* Where should the prime numbers go in the divisibility diagram?

*Problem 6.* Prove that a positive integer  $n$  is prime if and only if  $n$  is not divisible by any prime  $p$  with  $1 < p \leq \sqrt{n}$ .

*Problem 7.* Suppose that a positive integer  $n$  has prime factorization  $n = p_1 p_2 \cdots p_k$ . How many distinct positive integers are divisors of  $n$ ?

*Problem 8.* The book's proof does a fine job of guaranteeing that prime factorizations of integers are unique, but it elides the proof that prime factorization *exist*. Give an inductive proof that every positive integer has a prime factorization.