MATH 113: DISCRETE STRUCTURES
FINAL EXAM REVIEW

» Trees
- Definition: connected acyclic graph.
- Trees have leaves.
- Trees can be grown.
- Trees with n vertices have $n-1$ edges.
- Cayley's theorem: The number of labeled trees with n vertices is n^{n-2}.
- Prüfer codes.

» Catalan numbers
- Catalan number C_n counts unlabelled full binary trees with $n+1$ leaves.
- Catalan recurrence: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$.
- Closed formula: $C_n = \frac{1}{n+1}\binom{2n}{n}$.
- Other structures counted by Catalan numbers:
  - ∗ Dyck paths.
  - ∗ Parenthesizations.
  - ∗ Temperley–Lieb diagrams.

» Probability theory
- Basic objects: sample space, events, probability distribution.
- Uniform distribution on a finite sample space.
- Independent events.
- Conditional probability $P(A|B)$.
- Bayes' Law: $P(B|A) = P(A|B)P(B)/P(A)$.
- Law of Total Probability: $P(A) = \sum_i P(A|B_i)P(B_i)$ for $\{B_i\}$ a partition.
- Random variables.
- Expected value $E(X)$.
- Linearity of expected value: $E(cX+Y) = cE(X) + E(Y)$.
- Special types of random variables:
  - ∗ Bernoulli.
  - ∗ Binomial.
  - ∗ Indicator.

» Number theory
- Divisibility.
- Prime numbers.
- Fundamental theorem of arithmetic (unique prime factorization).
- Infinitude of primes.
- Prime number theorem: $\pi(n) \sim n/\log n$.
- Fermat's little theorem: $p|a^p - a$.
- Greatest common divisors and Euclidean algorithm.
- Congruences.
- Multiplicative inverses modulo n.
- Euler ϕ function: $\phi(n) = |\{r \in \mathbb{N} \mid r < n \text{ and } \gcd(r,n) = 1\}|$.
- Euler's formula: $\phi(n) = n(1-1/p_1)(1-1/p_2)\ldots(1-1/p_k)$ for n with prime factors $p_1, \ldots, p_k$.

Problem 1. Suppose that the vertices of a graph have degrees $4, 1, 1, 1, 1$. Decide whether this graph is a tree, is not a tree, or could be a tree or non-tree and prove your assertion. What if the vertex degrees are $3, 3, 2, 1, 1$?

Problem 2. Suppose that $2n$ people are seated around a circular table. In how many ways can they simultaneously shake hands with another person at the table so that none of their arms cross each other? Draw pictures of the $n = 1, 2, 3$ cases, come up with a conjecture, and prove it.

Problem 3. For a permutation σ of $\underline{2n}$, let $X(\sigma)$ be the number of $i \in \underline{2n}$ such that $\sigma(i) > 2i$. Determine the expected value of X.

Problem 4. For $a, b, c, n \in \mathbb{Z}$ suppose that $ac \equiv bc \pmod{n}$. Let $d = \gcd(c, n)$ and prove that

$$a \equiv b \pmod{n/d}.$$