MATH 113: DISCRETE STRUCTURES
DERANGEMENTS

Imagine n suitors all trying to woo each other. They each purchase a bouquet of flowers, and proceed to give the bouquet to their beloved. Assume further that the suitors are in the fortunate situation that no two suitors have the same beloved, and also that no suitor is so narcissistic as to have themself as beloved. In how many ways might the suitors distribute their bouquets?

We rephrase this problem mathematically as follows: an assignment of bouquets is a function $f : \underline{n} \to \underline{n}$ where $\underline{n} = \{1, 2, \ldots, n\}$. Since now two suitors have the same beloved, the function is injective, and thus surjective as the domain and codomain have the same cardinality. Thus f is a permutation. Finally, the non-narcissism clause guarantees that $f(i) \neq i$ for all $i \in \underline{n}$. When $f(i) = i$, we call i a *fixed point* of f, so we are looking for permutations of $\underline{n}$ with no fixed points. Such permutations are called *derangements*. The problem of enumerating derangements was first posed by Pierre de Montmort in 1708, and subsequently resolved independently by de Montmort and Nicholas Bernoulli in 1713.

The number of derangements of $\underline{n}$ is called the *subfactorial* of n and is denoted $n¡$. (Other notations include $!n$, $D(n)$, and D_n, but we will use the inverted exclamation point. While typically used at the start of exclamatory Spanish-language sentences, in 1668, John Wilkins proposed the punctuation ¡ at the end of a sentence to denote irony.)

In order to count $n¡$, we will count the "bad" permutations of $\underline{n}$ with at least one fixed point. For $i \in \underline{n}$, let A_i denote the set of permutations of $\underline{n}$ with i as a fixed point. Then $n¡ = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$. We aim to count $|A_1 \cup \cdots \cup A_n|$ via the inclusion-exclusion principle.

Note that $|A_i| = (n-1)!$. Indeed, for $f \in A_i$, $f(i) = i$ and f is free to permute the other $n-1$ elements of $\underline{n}$. What about $|A_i \cap A_j|$, $i \neq j \in \underline{n}$? If $f \in A_i \cap A_j$, then $f(i) = i$ and $f(j) = j$, but f is free to permute the other $n-2$ elements of $\underline{n}$, so $|A_i \cap A_j| = (n-2)!$. Similarly, if $i_1 < i_2 < \cdots < i_k$, then $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$. Since there are $\binom{n}{k}$ k-fold intersections, and each has the same cardinality $(n-k)!$, inclusion-exclusion implies that

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}.$$

Thus

$$n¡ = n! - |A_1 \cup \cdots \cup A_n| = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}.$$

By factoring out $n!$ (and replacing 1 with $\frac{1}{0!}$), we can further rewrite this as

$$n¡ = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

This gives us a formula for the number of derangements of $\underline{n}$, and also a count for our initial problem regarding distribution of bouquets.

If you have taken calculus, you may recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus when $n \to \infty$,

$$\frac{n¡}{n!} \longrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \approx 0.36788.$$

The convergence of this series is quite rapid, and it is actually the case that $n¡$ is the integer nearest $n!/e$ for all $n > 0$.

This is quite remarkable! About 0.36788 of permutations of $\underline{n}$ are derangements, independent of n. For reference, here is a table listing n, n¡, and the approximate value of $n!/e$.

n	n¡	$n!/e$
1	0	0.36788
2	1	0.73576
3	2	2.20723
4	9	8.82911
5	44	44.1455
6	265	264.873
7	1854	1854.11
8	14833	14832.9