MATH 113: DISCRETE STRUCTURES
FINAL EXAM REVIEW

The final is a three-hour exam that will be distributed via email at 9A.M. Pacific on Monday, December 14, and is due via Gradescope by 11P.M. Pacific on Tuesday, December 15. During this window, set aside an uninterrupted block of time to take the exam. During the exam, you may reference the text (*Discrete Structures* by Ormsby & Perkinson), your notes, your homework assignments, and the group problems and solutions. All work, including typesetting in LaTeX, should be done during the three-hour period (this does not include scanning and uploading to Gradescope).

- Counting fundamentals
  - Additive, multiplicative, and over-counting principles
  - Factorials, binomial coefficients, Pascal's triangle
  - Binomial theorem
- Equivalence relations
  - Reflexive, symmetric, transitive
  - Equivalence relations on S are in bijection with partitions of S (by taking eequivalence classes)
  - Relation to overcounting principle
- Induction proofs
  - Base case $P(0)$ + induction step $P(n) \implies P(n+1)$
- Inclusion/exclusion
  - Generalization of $|A \cup B| = |A| + |B| - |A \cap B|$ to union of many sets
- Pigeonhole principle
  - Contrapositive to "if $f\colon A \to B$ is injective, then $|A| \le |B|$."
- Recurrence relations and difference operators
  - For f a polynomial, $f(x) = \sum_k \Delta^k[f](0) \cdot \binom{x}{k}$
  - Calculate via table of differences
  - Numerical polynomials can be uniquely expressed as $\sum_k a_k \binom{x}{k}$ for $a_k \in \mathbb{Z}$
- Generating functions
  - For sequence (a_n), generating function $a(x) := \sum_n a_n x^n$
  - Recurrence relations create functional equations; solve $a(x)$ and use known series to deduce a closed formula for a_n
- Graph theory
  - Vertices, edges, degree
  - Handshake theorem: $\sum_{v \in V} \deg_G(v) = 2|E|$
  - Paths, cycles, theorem about existence of Eulerian walks/circuits
  - Trees
    - Definition: connected acyclic graph
    - Trees with n vertices have $n-1$ edges

    * Cayley's formula: the number of labeled trees with n vertices is n^{n-2} (proof via vertebrates)
- Catalan structures
  - Catalan number C_n counts unlabelled full binary trees with $n+1$ leaves
  - Catalan recurrence: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$
  - Closed formula: $C_n = \frac{1}{n+1}\binom{2n}{n}$
  - Structures counted by Catalan numbers:
    * Dyck paths
    * full binary trees
    * balanced parenthesizations
    * parethesizations of binary operators
    * noncrossing partitions
    * increasing parking functions
- Probability theory
  - Basic objects: sample space, events, probability distribution
  - Uniform distribution on a finite sample space
  - Independent events: $P(A \cap B) = P(A)P(B)$
  - Conditional probability: $P(A|B) = P(A \cap B)/P(B)$
  - Bayes' Law: $P(B|A) = P(A|B)P(B)/P(A)$
  - Law of Total Probability: $P(A) = \sum_i P(A|B_i)P(B_i)$ for $\{B_i\}$ a partition
  - Random variables
  - Expected value $E(X)$
  - Linearity of expected value: $E(cX + Y) = cE(X) + E(Y)$
  - Special types of random variables: Bernoulli, binomial, indicator
- Number theory
  - Divisibility
  - Prime numbers
  - Fundamental theorem of arithmetic (unique prime factorization)
  - Infinitude of primes
  - Prime number theorem: $\pi(n) \sim n/\log n$
  - Fermat's little theorem: $p \mid a^p - a$, i.e., $a^p \equiv a \pmod{p}$.
  - Greatest common divisors and Euclidean algorithm
    * Compute $\gcd(a, b)$ via Euclidean algorithm
    * "Backtrack" or use the extended Euclidean algorithm to solve for $s, t \in \mathbb{Z}$ in Bézout's identity: $\gcd(a, b) = as + bt$
  - Congruences / modular arithmetic
  - Units modulo n: The integer a has a multiplicative inverse modulo n if and only if $\gcd(a, n) = 1$.
  - Euler totient function φ
    * $\phi(n) = |\{r \in \mathbb{N} \mid r < n \text{ and } \gcd(r, n) = 1\}| = |\mathbb{Z}/n\mathbb{Z}^\times|$
    * Euler's formula: $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2)\ldots(1 - 1/p_k)$ for n with prime factors $p_1, \ldots, p_k$
    * Euler's theorem: if $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$
  - Sunzi's theorem: be able to state the theorem and to compute solutions

Links to compiled lists of homework and group problems:

- https://people.reed.edu/~davidp/113/handouts/compiled-hw.pdf
- https://people.reed.edu/~davidp/113/handouts/compiled-grp.pdf
- https://people.reed.edu/~davidp/113/handouts/compiled-grp-sol.pdf