MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 15

Problem 1. Consider the sequence (a_n) defined by

$$a_0 = 0, \qquad a_n = 2a_{n-1} + n \text{ for } n > 0.$$

(a) Compute $a_0, a_1, \ldots, a_{10}$.

(b) (No work required.) If you have taken calculus, convince yourself that

$$\sum_{n=0}^{\infty} nx^n = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{x}{(1-x)^2}.$$

(c) Use the defining recurrence to show that the generating function for (a_n) is

$$a(x) = \frac{x}{(1-2x)(1-x)^2}.$$

Hint: The identity from (b) should figure in your work.

(d) Use algebra to prove that

$$a(x) = \frac{2}{1-2x} - \frac{1}{1-x} - \frac{1}{(1-x)^2}.$$

(Look into the method of *partial fractions* if you want to learn how to generate such identities yourself.)

(e) (No work required.) If you have taken calculus, convince yourself that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(n+1)x^n.$$

(f) Use parts (d) and (e) to give a closed formula for a_n.

Problem 2. Define a sequence of Fibonacci-like numbers L_n by

$$L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-1} + L_{n-2} & \text{if } n > 1. \end{cases}$$

(a) Compute $L_0, L_1, \ldots, L_{10}$.

(b) Use (strong) induction to show that

$$L_n = F_{n-1} + F_{n+1}$$

for $n \geq 0$, using the convention that $F_{-1} = 1$. (This is a good convention since it satisfies $F_{-1} + F_0 = F_1$.)

(c) Determine the generating function for L_n as a rational function (*i.e.* as a quotient of two polynomials).

(d) Prove that

$$L_n = \phi^n + \overline{\phi}^n.$$

(e) Prove that

$$F_{2n} = F_n L_n$$

for $n \geq 0$.