MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 8

Your reading focused on *noncrossing partitions*. What about partitions in general, with no restriction on crossing? The following problems answer some combinatorial questions related to these structures.

Problem 1. Determine all of the partitions of the sets $\varnothing = [0], [1], [2], [3]$. Which partitions of $[4]$ cross (*i.e.*, which partitions are not noncrossing)?

Problem 2. Let B_n denote the number of partitions of $[n]$; this is called the n-th *Bell number*.

(a) Prove that for $n \geq 0$,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

(*Hint*: Removing the block containing $n+1$ from a partition of $[n+1]$ leaves a partition of k elements for some $0 \leq k \leq n$.)

(b) Use this recurrence and Problem 1 to determine B_4 and B_5.

Problem 3. For $k, n \geq 0$, the *Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is defined to be the number of partitions of $[n]$ into k blocks. Prove that

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0, \quad \text{and} \quad \left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$$

for $k, n > 0$. Use this to compute the Stirling numbers with $0 \leq k, n \leq 5$. (*Note*: The relation $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ will allow you to check your work.)

Remark. Famously, there is no "nice" formula for B_n. It is known (but you are not asked to prove) that

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

You can plug this into $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ to get a rather un-nice closed formula for the Bell numbers.