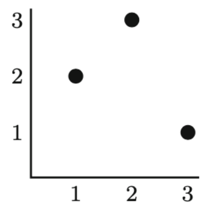
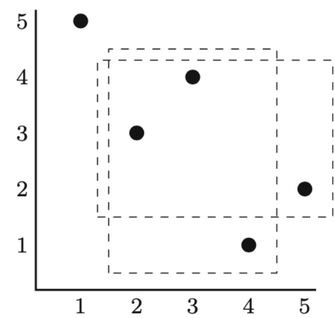
MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 7

A permutation $\pi \in \mathfrak{S}_n$ is called *231-avoiding* if there is no $1 \le i < j < k \le n$ such that $\pi(k) < \pi(i) < \pi(j)$. In other words, there are no $i < j < k$ such that $\pi(i), \pi(j), \pi(k)$ are in the same relative positions as $2, 3, 1$. It is easiest to check this condition visually with the graph (in the sense of functions) of the permutation. For instance, the permutation 53421 contains the pattern 231 in two ways, and thus is not 231-avoiding:





Let $\mathfrak{S}_n(231)$ denote the set of 231-avoiding permutations in $\mathfrak{S}_n$; let $s_n(231) := |\mathfrak{S}_n(231)|$.

Problem 1. List all of the 231-avoiding permutations for $n = 1, 2, 3, 4$ and compute the associated values of $s_n(231)$.

Problem 2. In this problem, you will prove that $s_n(231) = C_n$, the n-th Catalan number, by checking that these numbers satisfy the Catalan recurrence (Proposition 116 in the text).

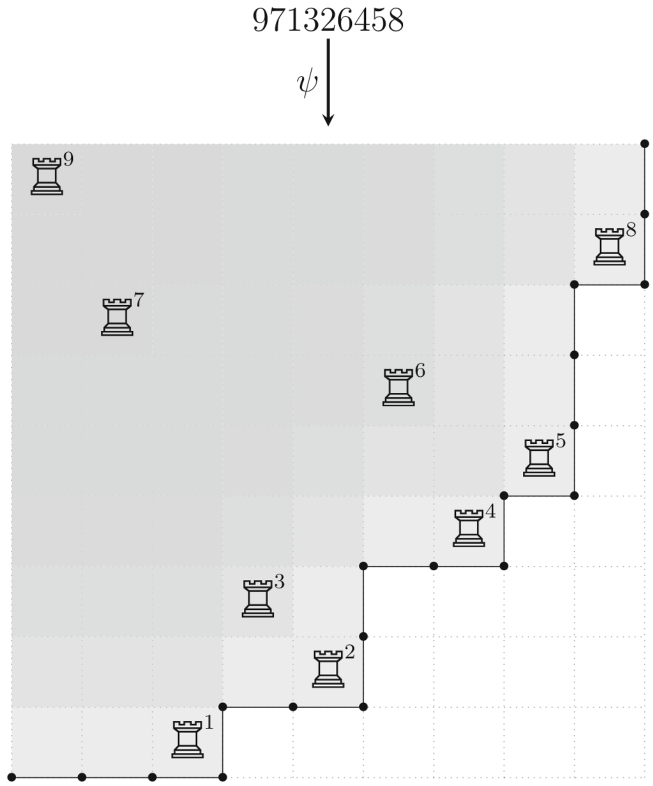
(a) Suppose that $u \in \mathfrak{S}_i(231)$ and v is a permutation of $\{i+1, \dots, n-1\}$ that avoids 231. Prove that
$$\pi = u(1) \cdots u(i) n v(i+1) \cdots v(n-1)$$
is a 231-avoiding permutation and use this to argue that
$$\sum_{i=0}^{n-1} s_i(231) s_{n-1-i}(231) \le s_n(231).$$

(b) Suppose that $\pi \in \mathfrak{S}_n$ is 231-avoiding with $\pi(i+1) = n$. Observe that the permutations $u = \pi(1) \cdots \pi(i)$ and $v = \pi(i+2) \cdots \pi(n)$ are 231-avoiding. Prove that all the values in u are smaller than all the values in v and use this to show that
$$s_n(231) \le \sum_{i=0}^{n-1} s_i(231) s_{n-1-i}(231).$$

(c) Check the base case and use induction and Proposition 116 to prove that
$$s_n(231) = C_n.$$

Problem 3. We now exhibit a direct bijection between 231-avoiding permutations and Dyck paths of length $2n$, providing another proof that $s_n(231) = C_n$.

(a) Given $\pi \in \mathfrak{S}_n(231)$, think of the graph of π as a configuration of rooks on an $n \times n$ chess board. Shade all the squares that either contain a rook or are to the left of or above a rook. Let $\psi(\pi)$ denote the bottom-right boundary of the shaded region and prove that $\psi(\pi)$ is a Dyck path of length $2n$.



Hint: In fact, every $\pi \in \mathfrak{S}_n$ produces a Dyck path in this fashion.

(b) Show that the function ψ from $\mathfrak{S}_n(231)$ to Dyck paths of length $2n$ is a bijection, and conclude that $s_n(231) = C_n$.