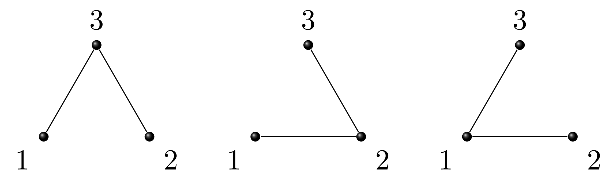
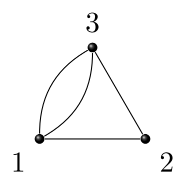
MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 7

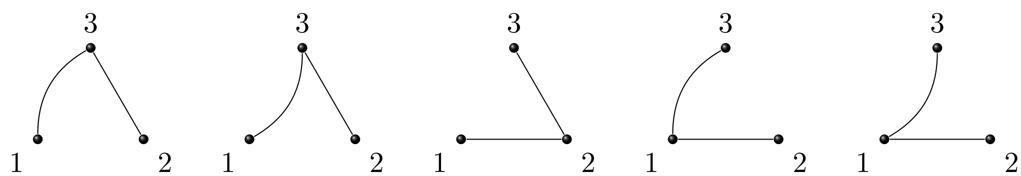
A *spanning tree* of a connected graph G is a subgraph T such that T is a tree and every vertex of G is on some edge of T. For instance, if G is the triangle with vertices $1, 2, 3$, then its spanning trees are:



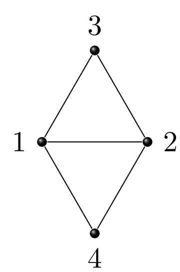
Recall that a *multigraph* is a graph in which multiple edges are allowed. For instance, the following graph has two edges connecting the vertices 1 and 3:



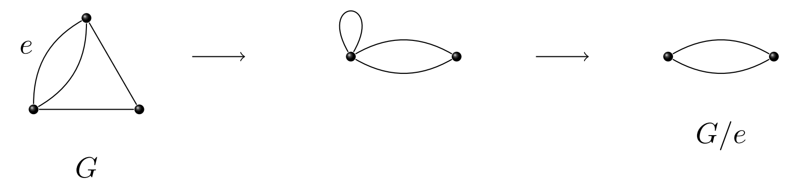
It has five spanning trees:



Problem 1. Draw all spanning trees of the following graph:



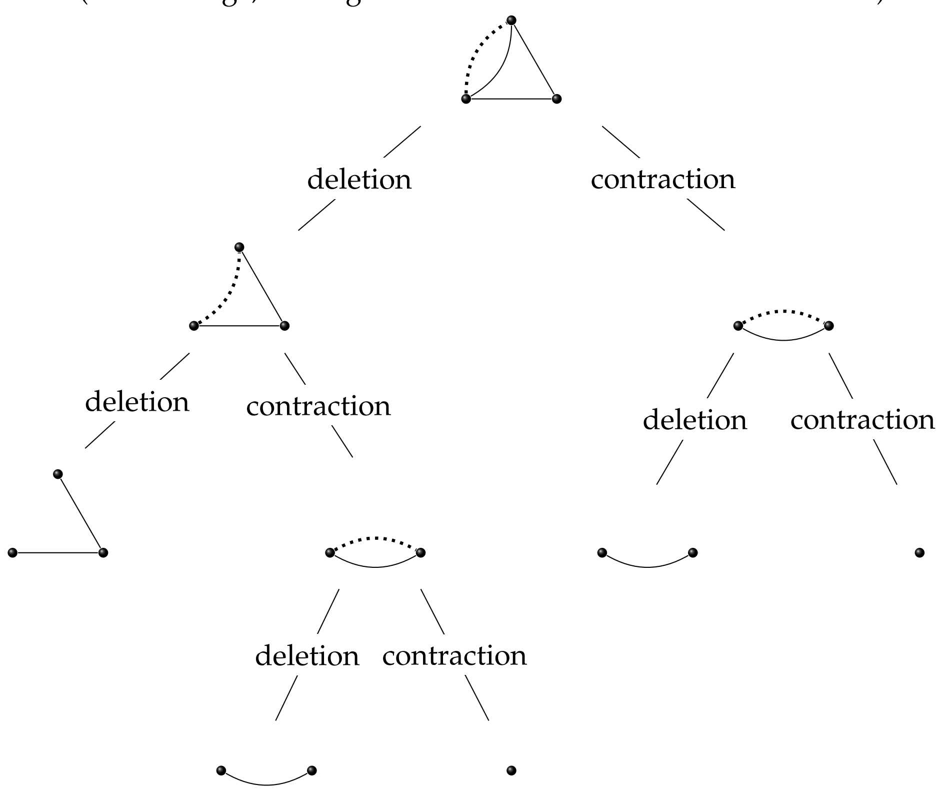
Problem 2 (deletion and contraction). Let G be a multigraph, and let e be an edge of G. Define $G - e$ to be the graph obtained from G by removing the edge e (but retaining the endpoints of e). Let G/e be the graph obtained from G by "contracting" the edge e. To contract e, remove e from G and then glue the endpoints of e together to make a single vertex from the two vertices. If there were multiple edges between the endpoints of e, loops will be formed, but for our purposes, we remove these loops as in the following:



(a) For an arbitrary connected multigraph G, choose an edge e such that $G - e$ is connected. Let $T(G)$, $T(G-e)$, and $T(G/e)$ denote the number of spanning trees of G, $G-e$, and G/e, respectively. The spanning trees of G come in two types: those that contain e and those that do not. Use that idea to prove

$$T(G) = T(G-e) + T(G/e).$$

(b) We can use the previous problem iteratively to count spanning trees. This is illustrated in the diagram below (at each stage, the edge chosen to delete and contract is dotted):



We stop in this deletion-contraction process when there are no edges left whose removal would leave a connected graph. Along the bottom, there are 5 trees (a single isolated vertex is considered to be a tree, too). The previous part of this problem implies there are 5 spanning trees of the original graph. These are the 5 spanning trees we saw earlier.

Make a similar diagram for the graph in Problem 1. (This diagram should verify the number of spanning trees you found earlier.)[1]

[1] The first part of Problem 2 implies the amazing fact that number of trees at the bottom of the diagram is independent of the choices of edges made in constructing the diagram!