MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 2

Problem 1. Let A and B be finite sets. Explain why

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Problem 2. Let A and B be sets with cardinalities $|A| = m$ and $|B| = n$. Suppose that $m \leq n$.

(a) What are the maximal and minimal values of $|A \cup B|$, and under what circumstances are these values achieved?

(b) What are the maximal and minimal values of $|A \cap B|$, and under what circumstances are these values achieved?

Problem 3. We have seen that there are 2^n subsets of a set A of cardinality n. We can use an *n-bit string* to encode such a subset. This is a length n word in the alphabet $\{0, 1\}$. Such an object looks like $b_{n-1}b_{n-1}\ldots b_0$ where each $b_i \in \{0, 1\}$, $0 \leq i \leq n-1$. To turn a subset into a bit string, label the elements of A as $A = \{a_0, a_1, \ldots, a_{n-1}\}$; then for $B \in 2^A$, set

$$b_i = \begin{cases} 1 & \text{if } a_i \in B, \\ 0 & \text{if } a_i \notin B. \end{cases}$$

For instance, if $A = \{0, 1, 2, 3\}$ and $B = \{0, 2, 3\}$, then the associated bit string is 1101.

Given a bit string, we may treat it as a *binary representation* of a number. This associates the number

$$[b_{n-1}b_{n-2}\ldots b_1b_0]_2 = b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_1 2^1 + b_0 2^0$$

with the bit string $b_{n-1}\ldots b_1b_0$. In the case of the bit string 1101, we have

$$[1101]_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 13.$$

(Of course, the final expression is a *decimal representation*: $13 = 1 \cdot 10^1 + 3 \cdot 10^0$.)

By turning a subset into a bit string and then a bit string into a number, we get a one-to-one correspondence between the subsets of A and the integers $0, 1, \ldots, 2^n - 1$. The following questions all refer to this numerical encoding of subsets.

(a) What numbers correspond to subsets of cardinality one?

(b) What number corresponds to the subset $A \in 2^A$?

(c) What subsets correspond to even numbers?