PROBLEM 1.

(i) Let $a \in \mathbb{Z}$ have digits $a_k a_{k-1} \ldots a_1 a_0$. In other words,

$$a = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k.$$

For example, $547 = 7 + 4 \cdot 10 + 5 \cdot 10^2$. Show that

$$a = a_0 + a_1 + \cdots + a_k \bmod 9.$$

(ii) Show how the above observation allows you to easily check that the integer 12345678 is divisible by 9.

The observation in Problem 1 is the basis of the common and useful way of checking arithmetic called "casting out 9s". To check that your arithmetic is correct in adding a collection of multi-digit integers, first add all the digits, casting out 9s as you go to keep the sum small. The result is the sum of the digits modulo 9. Next add the digits of your answer, again casting out 9s. If your two results don't agree modulo 9, then you made an arithmetic mistake somewhere. For instance, consider the following calculation:

$$\begin{array}{rr} & 59284 \\ + & 27968 \\ \hline & 86252 \end{array}$$

Working modulo 9,

$$\begin{aligned} 59284 &= 5+9+2+8+4 = (5+4)+9+2+8 \\ &= 0+0+2+(-1) = 1 \end{aligned}$$

$$27968 = 2+7+9+6+8 = (2+7)+9+6+(-1) = 0+0+6-1 = 5.$$

Therefore,

$$59284 + 27968 = 1 + 5 = 6 \bmod 9.$$

On the other hand, again modulo 9,

$$86252 = 8+6+2+5+2 = -1+6+(2+5+2) = -1+6+9 = -1+6 = 5.$$

So

$$59284 + 27968 = 6 \neq 5 = 86252 \bmod 9,$$

which shows the arithmetic is faulty.

The digits of 59284 and 27968 were processed separately above, but they could have been combined: $5+9+2+8+4+2+7+9+6+8 =$ etc. mod 9, looking for pairs adding to 9 to discard.

PROBLEM 2.

(i) Apply the method of casting out 9s to show the following arithmetic is mistaken. (As you go, look for digits that sum to 9 casting these out since that don't effect the sum modulo 9.)

$$\begin{array}{rr} & 183 \\ & 247 \\ & 346 \\ & 739 \\ + & 435 \\ \hline & 1960 \end{array}$$

What is the sum of the numbers modulo 9 (above the line), and what is the (incorrect) bottom-line sum modulo 9?

(ii) Explain why the casting out 9s method of error-checking is not foolproof. Give a concrete example.

PROBLEM 3.

(i) Show that if the digits of $a \in \mathbb{Z}$ are $a_k a_{k-1} \dots a_1 a_0$, then

$$a = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k \bmod 11.$$

(Note that this gives a method for determining whether an integer is divisible by 11.)

(ii) Find a number between 0 and 10 that is 12345678987654321 mod 11 by using the above trick.

PROBLEM 4. Consider a sample space S consisting of the set of *even* integers n such that $0 \leq n \leq 98$, and give S the uniform probability distribution. Each element of S may be thought of as a two digit number $n = d_2 \cdot 10 + d_1$.

(i) What is the probability that $d_1 \neq d_2$ for a randomly drawn element from S.

(ii) Let X be the random variable $X(n) = d_1 + d_2$ for $n = d_2 \cdot 10 + d_1$. What is the expected value $E(X)$?

(iii) Let $A = \{n \in S : X(n) = 8\}$ and $B = \{n \in S : n < 50\}$. Are A and B independent events?

(iv) With A and B as above, what is $P(A|B)$?