PROBLEM 1. Let NC_n denote the number of noncrossing partitions of $[n]$. In the text, you saw a direct bijection exhibiting that $NC_n = C_n$. Reprove this via the Catalan recurrence:

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \text{ for } n \geq 0.$$

Hint: For the inductive step, consider any noncrossing partition P of $[n+1]$. The number $n+1$ is in some block X of P. Let k be the next largest number in X, or set $k = 0$ if $X = \{n+1\}$. Observe that in the partition P, every part contains either only numbers bigger than k or only numbers smaller than k. (Why?)

PROBLEM 2.

(i) Following the tips at the end of the video lecture, formulate the direct bijection between Dyck paths of length $2n$ and noncrossing partitions of $[n]$.
(ii) Call a transition from an east step to a north step in a Dyck path a *valley*. Verify that the number of valleys in a Dyck path corresponds to the number of blocks in the associated partition.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Show that the number of noncrossing partitions of $[n]$ with exactly k blocks is the *Narayana number*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

This is also the number of Dyck paths of length $2n$ with exactly k valleys. Conclude that

$$C_n = \sum_{k=1}^{n} N(n,k).$$