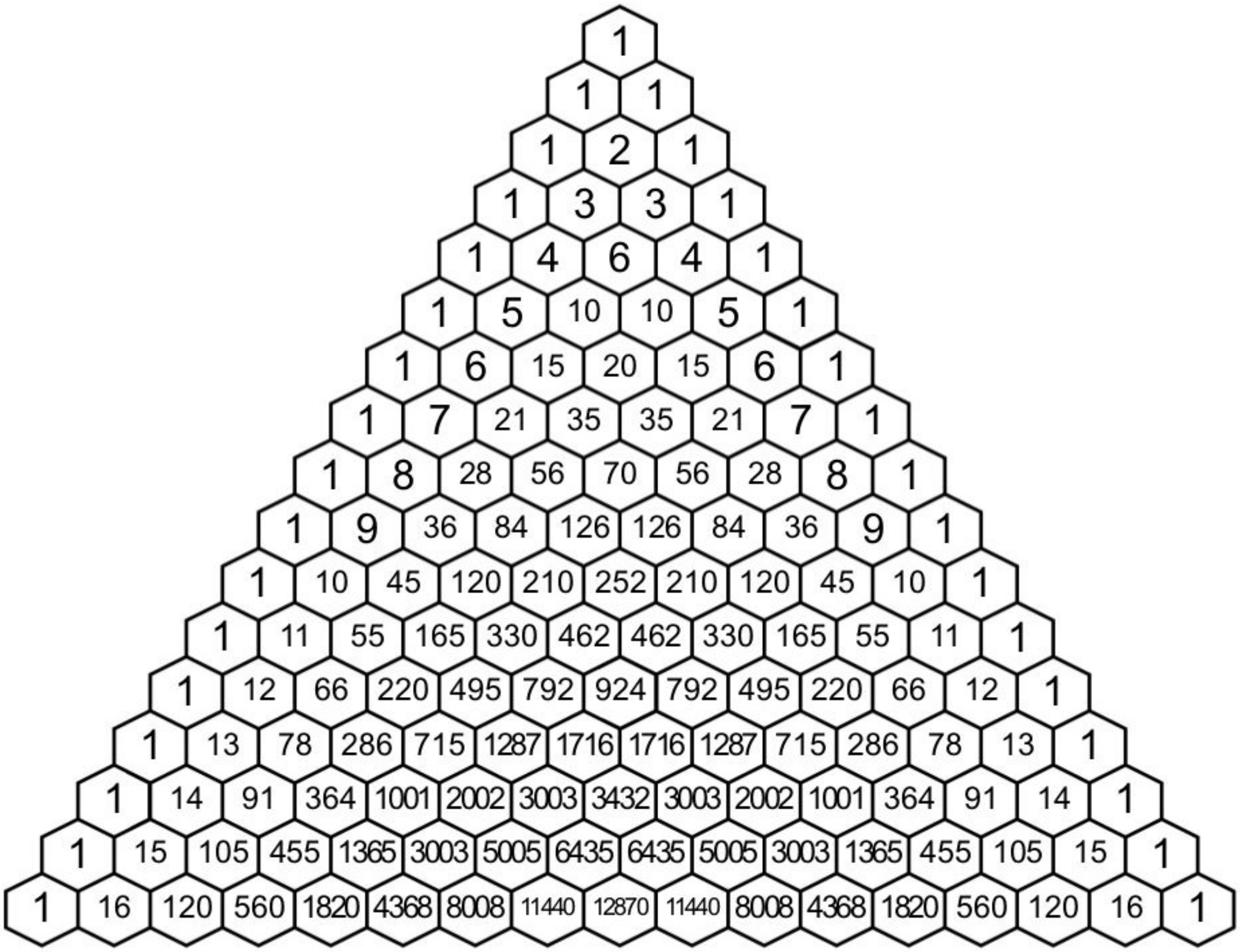
For reference, here is a copy of Pascal's triangle:



PROBLEM 1. The book claims that

$$\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{n+1}{k+1}$$

for all $k, n \in \mathbb{Z}$.

(i) Highlight the terms involved in this identity for various k and n on Pascal's triangle; explain why it is known as the *hockey stick identity*.

(ii) Let X be the set of subsets of $[n+1]$ of cardinality $k+1$, and let

$$X_a := \{A \in X \mid a \text{ is the first element of } [n+1] \text{ in } A\}$$

for $a = 1, 2, \ldots, n-k$. Check that

$$X = X_1 \amalg X_2 \amalg \cdots \amalg X_{n-k+1}.$$

(iii) Determine the cardinality of X_a in terms of n, k, and a. Use this and (ii) to give a combinatorial proof of the hockey stick identity.

PROBLEM 2.

(i) Compute the sums

$$\binom{0}{0}^2$$
$$\binom{1}{0}^2 + \binom{1}{1}^2$$
$$\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$
$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$$
$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$
$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

(ii) Use the binomial theorem to prove your conjecture. [*Hint*: Consider the coefficient of x^n in $(1+x)^{2n} = (1+x)^n(1+x)^n$.]

(iii) Give a combinatorial argument proving your conjecture. [*Hint*: Split a set of size $2n$ into two pieces of size n, and then start building size n subsets of the original set.]

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

How many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.) Develop a conjecture and prove it.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Answer the variation of Problem 3 in which we allow *nonnegative* integer summands.