MATH 113: DISCRETE STRUCTURES
EQUIVALENCE RELATIONS

Consider the problem of putting King Arthur and his twelve knights in a line. Thirteen different people can take the first spot in line, twelve can take the second, *etc.*, until there is only one person who can take the final spot. We deduce that there are

$$13 \cdot 12 \cdot 11 \cdots 2 \cdot 1 = 13!$$

ways for the heroes of Camelot to queue up.

Note, though, that Arthur and his knights are famous enough that they rarely have to wait in line. With the extra leisure time this affords, they like to sit at the Round Table. Since the table is round, we consider seatings to be "the same" or "equivalent" if one can be rotated to produce the other. (Rotation by $0°$ counts, so any given seating is equivalent to itself.)

With this notion of rotational equivalence in hand, we can break up the queuings of the first paragraph into "equivalence classes" of seatings that can be rotated into each other. Since each such equivalence class consists of 13 lineups, there are a total of

$$13!/13 = 12!$$

seatings that cannot be rotated into each other.

Our task in these notes is to formalize the above ideas and see how they fit into combinatorics.

1. Definitions and examples

Definition 1.1. A *relation* R on a set A is a subset of $A \times A$. We write aRb when $(a, b) \in R$.

The idea here is to think of a being Related (somehow) to b when aRb, *i.e.*, when $(a, b) \in R$. It is also common to use a special symbol such as $\sim$, $\simeq$, $\cong$, or $\equiv$ to denote a relation. The particular symbols just mentioned are more common when the relation is in fact an equivalence relation, which we presently define.

Definition 1.2. A relation $\sim$ on A is an *equivalence relation* if it is

(a) *reflexive*: for all $a \in A$, $a \sim a$,
(b) *symmetric*: for $a, b \in A$, if $a \sim b$, then $b \sim a$, and
(c) *transitive*: for $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Let S denote the set of students in a class. We can define an equivalence relation $\cong$ on S by declaring that $s \cong t$ if and only if s and t have the same birthday. Let's check that it forms an equivalence relation. Clearly for each $s \in S$, s has the same birthday as s, so $s \cong s$. If s has the same birthday as t, then t has the same birthday as s, so $s \cong t$ implies that $t \cong s$. Finally, if s has the same birthday as t and t has the same birthday as u, then s has the same birthday as u, so the relation is transitive. We conclude that $\cong$ is an equivalence relation on S.

Now consider the King Arthur problem again. To make life easier, let's number the Camelotians 1, 2, 3, ..., 13. Let Q denote the set of queues of $1, 2, \ldots, 13$, *i.e.*, the set of permutations of $\underline{13} = \{1, 2, \ldots, 13\}$. Two queues create the same seating if we can cyclically reorder (rotate the table) from one to the other, so we declare $q_1 \sim q_2$ when we can cycle q_2 into q_1. The reader may check that this forms an equivalence relation.

2. EQUIVALENCE CLASSES AND PARTITIONS

Definition 2.1. Let A be a set and let $\sim$ be an equivalence relation on A. For $a \in A$, the *equivalence class* of a, written $[a]_\sim$ (or just $[a]$ if $\sim$ is clear from context) is the set

$$[a]_\sim := \{b \in A \mid a \sim b\}.$$

In the King Arthur problem, if $q = (1, 2, \dots, 13)$, then $[q]_\sim$ is the set of permutations that can be rotated into q. For instance, $(2, 3, \dots, 13, 1) \in [q]_\sim$.

More generally, think of the elements of a set as the residents of an apartment complex. Declare two elements equivalent if they live together. Then the equivalence classes are naturally in bijection with the apartments in the apartment building: we can think of an equivalence class as the set of people inhabiting a particular apartment.[1] The following theorem sharpens this analogy.

Theorem 2.2. *If A is a set and $\sim$ is an equivalence relation on A, then for all $a, b \in A$*

(1) $a \in [a]$,
(2) if $a \sim b$, then $[a] = [b]$,
(3) if $a \not\sim b$, then $[a] \cap [b] = \varnothing$, and
(4) $\bigcup_{a \in A} [a] = A$.

Some comments on the notation are in order. First, $a \not\sim b$ simply means that (a, b) is not an element of $\sim$. Second, the indexed union $\bigcup_{a \in A} [a]$ may look intimidating, but it just means that we take the union of all the sets $[a]$ where a runs through A.

Proof. (1) Since $\sim$ is reflexive, $a \sim a$ and thus $a \in [a]$.

(2) Suppose $a \sim b$ and $c \in [a]$. Then, by definition, $a \sim c$. Furthermore, symmetry tells us that $b \sim a$. Thus transitivity (applied to $b \sim a$, $a \sim c$) implies that $b \sim c$, *i.e.*, $c \in [b]$. This proves that $[a] \subseteq [b]$. The reader may now write down a nearly identical proof that $[b] \subseteq [a]$, whence $[a] = [b]$.

(3) Suppose $a \not\sim b$. We must show that if $c \in [a]$, then $c \notin [b]$. Suppose for contradiction that $c \in [a]$ and $c \in [b]$. Then $a \sim c$ and $b \sim c$. By symmetry and transitivity, we learn that $a \sim b$, a contradiction. We conclude that if $a \not\sim b$, then $[a] \cap [b] = \varnothing$.

(4) Since each $[a]$ is a subset of A, we know that $\bigcup_{a \in A} [a] \subseteq A$. The opposite inclusion follows from (1): if $b \in A$, then $b \in [b]$, and thus $b \in \bigcup_{a \in A} [a]$ because $[b]$ is one of the terms in the indexed union.

□

Properties (3) and (4) of equivalence classes in Theorem 2.2 tell us that equivalence classes form a "partition," a concept which deserves its own definition.

Definition 2.3. A family of subsets $P_i \subseteq A$, where i ranges through an index set I, is a *partition* of A if $i \neq j \in I$ implies that $P_i \cap P_j = \varnothing$ and $\bigcup_{i \in I} P_i = A$.

Going back to our apartment complex analogy, we have a set of residents in the building A and then sets P_i of residents in apartment i for each $i \in I$, where I is the set of apartments.

We have seen that an equivalence relation on a set A produces a partition of A into equivalence classes. The converse is true as well: each partition produces an equivalence relation on A.

Theorem 2.4. *Suppose $\mathcal{P} = \{P_i \subseteq A \mid i \in I\}$ is a partition of A. Define a relation $\sim$ on A where $a \sim b$ if and only if there exists $P_i \in \mathcal{P}$ such that both a and b belong to P_i. Then $\sim$ is an equivalence relation.*

[1] This is true under mild hypotheses on the apartment building: every apartment has at least one resident, and no residents live in more than one apartment.

Proof. We first check that $\sim$ is reflexive. Given $a \in A$, we know that a is in some P_j, $j \in I$ because $\bigcup_{i \in I} P_i = A$. Thus $a \sim a$.

The definition of $\sim$ does not depend on the order of a and b, so $\sim$ is clearly symmetric: $a \sim b$ implies that $b \sim a$.

For transitivity, simply note that if both a and b are in P_i, and both b and c are in P_i, then a and c are in P_i. Thus $a \sim b$ and $b \sim c$ implies that $a \sim c$. ☐

The reader may check[2] that the constructions of this section give us a bijection between equivalence relations on A and partitions of A.

Since we are studying combinatorics in this class, it is only natural to ask how many partitions there are on A when $|A| < \infty$. This is a surprisingly subtle question, and we're not quite ready to develop the answer yet (but give it a try if you want to!).

3. Enumerating equivalence classes

Thinking about King Arthur's Round Table again, we see that we are trying to enumerate (count) the number of equivalence classes on Q, the set of queuings, with respect to the rotation equivalence relation $\sim$. The set of equivalence classes gets its own special notation: $Q/\sim$. We can reinterpret the argument from the introduction as saying that each equivalence class is of size 13. Thus the total number of equivalence classes is

$$|Q/\sim| = |Q|/13 = 13!/13 = 12!.$$

This is a general counting principle: If A is a set equipped with an equivalence relation $\sim$, and each of the $\sim$ equivalence classes has size m, then

$$|A/\sim| = |A|/m.$$

There is another way to count equivalence classes that we can again illustrate with the Round Table, namely, the method of choosing representatives. Suppose we have a way of picking exactly one representative from each equivalence class in $A/\sim$. Then the total number of such representatives will be equal to $|A/\sim|$. How can we do this for the Round Table problem? Well, since we can rotate the table, let's always put King Arthur at the top of it. Within each equivalence class of seatings, exactly one has Arthur at the top, so that will do the trick. Once we've put Arthur at the top, there are 12 ways to fill the seat to his left, then 11 ways to fill the left to the left of that one, *etc.*, revealing that there are

$$12 \cdot 11 \cdot 10 \cdots 1 = 12!$$

such representatives. We conclude that there are 12! seatings ($|Q/\sim|$) as well.

Let's do one more familiar example through the lens of equivalence relations. Consider the word $OUROBOROS$. (The ouroboros is an alchemical symbol for infinity in which a snake eats its own tail.) How many distinct strings can we make from the letters in $OUROBOROS$? We approach this by enumerating a larger set and then putting an equivalence relation on it so that the equivalence classes correspond to the distinct strings.

Let P be the set of permutations of the nine symbols $O_1, U, R_1, O_2, B, O_3, R_2, O_4, S$. We see that $|P| = 9!$. For $p, q \in P$, declare that $p \simeq q$ when p and q produce the same string after forgetting the subscripts. (For instance, $O_1O_2UO_3OR_1O_4R_2BS \simeq O_3O_4UO_2R_2O_1R_1BS$ because $OOUORORBS = OOUORORBS$.) If we can count $|P/\simeq|$, then we will have counted the number of distinct strings made from the letters in $OUROBOROS$. To this end, note that each equivalence class contains $4! \cdot 2! = 48$ permutations. (This is the number of ways to reorder the

[2]One of the most dangerous phrases in mathematical writing! You really should check when you see this, as it is too often a standin for "The author is too lazy to check."

four O's and two R's.) Thus our first counting principle tells us there are $|P/\simeq| = 9!/48 = 7560$ strings.