Lecture 5

Monday, February 2, 2015 7:55 AM

Summation and Induction

For integers m, n w/ $m \leq n$ and a function f defined on $m, m+1, m+2, \ldots, n$, we define

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n).$$

upper bound: end sum here.

Anatomy:

$$\sum_{k=m}^{n}$$

Greek letter Sigma (for "sum")

lower bound: start sum here

dummy variable

e.g. $f(k) = k$. Then $\sum_{k=-2}^{3} f(k) = f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3)$

$= -2 + (-1) + 0 + 1 + 2 + 3 = 3$

If we have a fmla for $f(k)$ (as above w/ $f(k) = k$) frequently write down fmla instead of $f(k)$:

e.g. $\sum_{k=-2}^{3} k$

e.g.

(1) $$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

(2) $$\sum_{k=2}^{4} (k^2 + k) = (2^2 + 2) + (3^2 + 3) + (4^2 + 4)$$

(3) $$\sum_{k=1}^{4} 3 = 3 + 3 + 3 + 3$$

Q If $a_0, a_1, a_2, \ldots, a_n$ are numbers & x is a variable, what is

$$\sum_{k=0}^{n} a_k x^k \ ?$$

A polynomial! $a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$

$$= a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

If $a_n \neq 0$, then this is a polynomial of degree n.

Q What do we make of $\sum_{k=m}^{n} f(k)$ when $n < m$?

This is the "empty sum" — it takes the value 0.

Products

$$\prod_{k=m}^{n} f(k) = f(m)\cdot f(m+1)\cdot f(m+2)\cdots f(n).$$

$\prod$ = Pi.

e.g. $$\prod_{k=1}^{n} k = 1\cdot 2\cdot 3 \cdots n = \underbrace{n!}_{\text{read as "n factorial"}}$$

Convention: empty products are equal to 1

In particular, $\prod_{k=1}^{0} k = 0! = 1$.

Mathematical Induction

Suppose $P(n)$ is a statement depending on integers n.

e.g. • $P(n)$: there exists a prime # larger than n

• $P(n)$: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is true

• $P(n)$: a certain property holds for all polynomials of degree $\leq n$

if $P(n) = T$ for all integers n then the property holds for all polynomials.

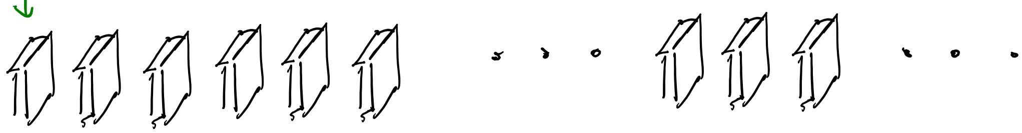
Suppose $P(n)$ is such a statement.

Then $P(n)$ is true for all $n \geq n_0$ if

① Base case: $P(n_0)$ is true

② Induction step: $P(m)$ true for $m = n_0, n_0+1, \ldots, n-1$ implies $P(n)$ true

topples by ①



$n_0 \quad n_0+1 \quad n_0+2 \qquad \qquad n-2 \quad n-1 \quad n$

② says if these fall, then they topple n as well.

① + ② are "strong induction".

① Base: $P(n_0)$ true

②′ $\forall n > n_0$, $P(n-1) \Longrightarrow P(n)$

induction: also implies $P(n)$ true $\forall n \geq n_0$.

<u>Proposition</u> $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \geq 1$.

<u>Pf by induction on n</u>: The base case is $n=1$. Then the LHS is $\displaystyle\sum_{k=1}^{1} k = 1$ and RHS is $\dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$, so $\displaystyle\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$.

For induction, assume that we have $n > n_0$ s.t.

$$\sum_{k=1}^{n-1} k = \frac{(n-1) \cdot ((n-1)+1)}{2}.$$

Then $$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k \right) + n \qquad \text{[algebra]}$$

$$= \frac{(n-1)((n-1)+1)}{2} + n \qquad \text{[induction hypothesis]}$$

$$= \frac{(n-1)\cdot n}{2} + \frac{2n}{2}$$

$$= \frac{n^2 - n + 2n}{2}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}. \qquad \text{[algebra]}$$

By induction we learn that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \geq 1$.

$\square$