Lecture 49

Friday, May 1, 2015 8:01 AM

$$F_1 = 1, \quad F_2 = 1 \longrightarrow F_i = F_{i-1} + F_{i-2} \text{ for } i \geq 3.$$

$F_3 = 2$

$F_4 = 3$

$F_5 = 5$

$F_6 = 8$

$F_7 = 13$

$F_8 = 21$

$F_9 = 34$

$F_{10} = 55$

$\vdots$

Generatingfunctionology

$$S = \sum_{i=1}^{\infty} F_i t^i$$

$$= t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 13t^7 + \cdots$$

$$= t + t^2 + \sum_{i=3}^{\infty} (F_{i-1} + F_{i-2}) t^i$$

$$= t + t^2 + \sum_{i=3}^{\infty} F_{i-1} t^i + \sum_{i=3}^{\infty} F_{i-2} t^i$$

$$= t + t^2 + t \sum_{i=3}^{\infty} F_{i-1} t^{i-1} + t^2 \sum_{i=3}^{\infty} F_{i-2} t^{i-2}$$

$$= t + t^2 + t \sum_{k=2}^{\infty} F_k t^k + t^2 \sum_{l=1}^{\infty} F_l t^l$$

$$\boxed{S = t + t^2 + t(S - t) + t^2 S}$$

$$\boxed{S = t + t^2 + t(S - t) + t^2 S}$$

$$S - t^2 S - tS = t + \cancel{t^2} - \cancel{t^2}$$

$$S(1 - t - t^2) = t$$

$$\boxed{S = \frac{t}{1 - (t + t^2)}}$$

(in whatever ball S converges within)

Let $\phi = \dfrac{1 + \sqrt{5}}{2} \approx 1.618\ldots$

$$\phi' = \frac{1 - \sqrt{5}}{2} \approx -0.618\ldots$$

Then $1 - (t + t^2) = (1 - \phi t)(1 - \phi' t)$

Friday, May 1, 2015 8:20 AM

so $S = \dfrac{A}{1-\phi t} + \dfrac{B}{1-\phi' t}$ for some A, B.

$$= \frac{A(1-\phi' t) + B(1-\phi t)}{(1-\phi t)(1-\phi' t)}$$

$$= \frac{(A+B) - (A\phi' + B\phi)t}{(1-\phi t)(1-\phi' t)}$$

$$\Longrightarrow A = -B, \quad -(A\phi' + B\phi) = 1$$

$$-A\phi' + A\phi = 1$$

$$A(\phi - \phi') = 1$$

$$A = \frac{1}{\phi - \phi'} = \frac{1}{\sqrt{5}}$$

$$\boxed{S = \frac{1/\sqrt{5}}{1-\phi t} - \frac{1/\sqrt{5}}{1-\phi' t}}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

for $|r| < 1$

$$\frac{1/\sqrt{5}}{1-\phi t} = \frac{1}{\sqrt{5}} \cdot \sum_{k=0}^{\infty} (\phi t)^k \quad \text{for } |\phi t| < 1$$

$$\frac{1/\sqrt{5}}{1-\phi' t} = \frac{1}{\sqrt{5}} \cdot \sum_{k=0}^{\infty} (\phi' t)^k \quad \text{for } |\phi' t| < 1$$

For both to converge, guarantee that

$$|t| < \frac{1}{|\phi|}, \frac{1}{|\phi'|}$$

i.e. $$|t| < \underbrace{\frac{1}{|\phi|}}_{} = -\phi' \approx 0.618$$

$$\frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)} = \frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{2(1-\sqrt{5})}{-4} = -\frac{1-\sqrt{5}}{2} = -\phi'.$$

So All manipulations w/ series above are valid for $|t| < -\phi'$!

$$S = \sum_{i=1}^{\infty} F_i t^i = \frac{1/\sqrt{5}}{1-\phi t} - \frac{1/\sqrt{5}}{1-\phi' t}$$

$(|t| < -\phi')$

$$= \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{\infty} (\phi t)^k - \sum_{k=0}^{\infty} (\phi' t)^k \right)$$

$$= \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{\infty} (\phi^k - \phi'^k) t^k \right)$$

$$= \frac{1}{\sqrt{5}} \sum_{i=1}^{\infty} (\phi^i - \phi'^i) t^i$$

$$\boxed{\sum_{i=1}^{\infty} F_i t^i = \sum_{i=1}^{\infty} \frac{1}{\sqrt{5}} (\phi^i - \phi'^i) t^i}$$

Thus $\boxed{F_i = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right)}$

for $i \geq 1$

p-adic numbers

p prime number

$\mathbb{Q}$ rational #s

$\left|\frac{a}{b}\right|_p = p^{-k}$ where $\frac{a}{b} = p^k \frac{a'}{b'}$ where $p \nmid a', b'$.

Analyze sequences in $\mathbb{Q}$ wrt metric induced by $|\cdot|_p$.

$\mathbb{Q}_p$ = p-adic #s = completion of $\mathbb{Q}$ wrt $|\ |_p$ metric.

In $\mathbb{Q}_p$, $\sum_{k=1}^{\infty} a_k$ converges $\iff \lim_{k\to\infty} a_k = 0$.