Lecture 47

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Now study $E(ix)$, $x \in \mathbb{R}$.

<u>Recall</u> $E(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

<u>Note</u>

$$(ix)^k = i^k x^k = \begin{cases} i x^k & \text{if } k \equiv 1 \pmod 4 \\ -x^k & \text{if } k \equiv 2 \pmod 4 \\ -i x^k & \text{if } k \equiv 3 \pmod 4 \\ x^k & \text{if } k \equiv 0 \pmod 4 \end{cases}$$

$$E(ix) = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!}$$

$$+ \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!}$$

$$+ \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \cdots$$

Re

Im

$$\mathrm{Re}\left(E(ix)\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = C(x)$$

$$\mathrm{Im}\left(E(ix)\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = S(x)$$

Observations

(0) $E(ix) = C(x) + iS(x)$

(1) $1 = E(0) = C(0) + iS(0)$

$$\Longrightarrow \quad C(0) = 1, \quad S(0) = 0.$$

(2) $C(-x) = C(x), \quad S(-x) = -S(x)$

$$E(-ix) = C(-x) + iS(-x)$$

$$= C(x) - iS(x)$$

$$= \overline{C(x) + iS(x)}$$

$$= \overline{E(ix)}$$

a^2+b^2
$= (a+bi)(a-bi)$

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(3) $C(x)^2 + S(x)^2$

$$= \big(C(x)+iS(x)\big)\cdot\big(C(x)-iS(x)\big)$$

$$= E(ix)\cdot E(-ix)$$

$$= E\big(ix+(-ix)\big)$$

$$= E(0)$$

$$= 1$$

Q Is $C = \cos$?
Is $S = \sin$?

$$\Rightarrow |E(ix)| = 1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow -1 \leq C(x), S(x) \leq 1.$$

(4) $\big(E(ix)\big)' = E'(ix)\cdot i = E(ix)\cdot i$

$$= i\,C(x) - S(x)$$

$\|$

$$\big(C(x)+iS(x)\big)' = C'(x) + i\cdot S'(x)$$

$$\Rightarrow \boxed{C'(x) = -S(x)}$$

$$\Rightarrow \boxed{\begin{aligned} C'(x) &= -S(x) \\ S'(x) &= C(x) \end{aligned}}$$

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Thm There exists unique $s \in (0, \sqrt{3})$ s.t.

$$E(is) = i \quad (\text{so } C(s) = 0,\ S(s) = 1).$$

Pf Idea ① Show $C(\sqrt{3}) < 0$

$C(0) = 1 > 0$

so IVT $\Rightarrow \exists\, s \in (0, \sqrt{3})$ s.t. $C(s) = 0$

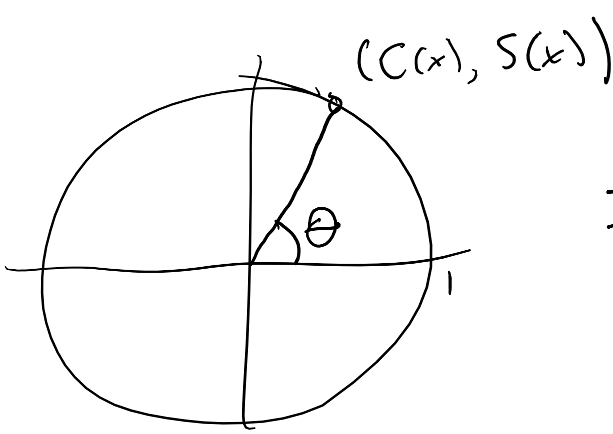
② $C^2 + S^2 = 1 \Rightarrow S(s) = \pm 1$

③ $S(t) > 0$ for $t \in (0, \sqrt{3}) \Rightarrow S(s) = 1$

④ C is strictly decreasing on $(0, \sqrt{3})$ and thus s is unique. $\square$

Thm $C = \cos$, $S = \sin$, $s = \pi/2$.

Pf Idea $E(ix) \longleftrightarrow (C(x), S(x))$ on unit circle



Is $\theta = x$???

Note that $\Theta(E(is)) = \Theta(i) = \frac{\pi}{2}$.

$S(s \cdot t) > 0$ for $t \in [0,1]$ $\Big\}$ Check via power series!

$\uparrow$ function of t for $t \in \mathbb{R}$

$(C(st))' = -s\,S(st) < 0$ for $t \in [0,1]$

so $C(st)$ is decreasing for $t \in [0,1]$

from $C(s\cdot 0) = C(0) = 1$ to $C(s \cdot 1) = C(s) = 0$.

Thus (b/c $C^2 + S^2 = 1$) we must have

$S(st)$ is increasing from 0 to 1 for $t \in [0,1]$.

$\Longrightarrow$ $E(ist)$ in first quadrant for $t \in [0,1]$.

Idea: Show angle of $E(ist)$ is $\frac{\pi}{2} t$ for t rational. Then use continuity to deduce that the same is true for all real t.

$m, n \in \mathbb{Z}, n \neq 0$ $\quad E(is \cdot m/n) = E(is \cdot 1/n)^m$

$\Longrightarrow$ $E(is\,m/n)$ has angle m times angle of $E(is/n)$.

Angle of $E(is/n)$ is angle of $E(is)^{1/n}$

which is $(\pi/2)/n$

$\Rightarrow$ angle of $E(is\, m/n)$ is $m\cdot(\pi/2)/n$

$$= \frac{\pi}{2}\cdot\frac{m}{n}$$

Thus for $t\in\mathbb{R}$, $E(is\cdot t)$ has angle $\frac{\pi}{2}\cdot t$.