Lecture 46

Monday, April 27, 2015 8:03 AM

A very important series

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

- $R = \infty$ so $E(x) \in \mathbb{C}$ $\forall x \in \mathbb{C}$
- $E(x)$ is differentiable on all of $\mathbb{C}$

Observations

(0) $E'(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)'$

$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

In fact, $\left(\frac{x^k}{k!}\right)' = \frac{k \cdot x^{k-1}}{k!}$

$$= \frac{\cancel{k}\, x^{k-1}}{\cancel{k} \cdot \underbrace{(k-1)(k-2)\cdots 2 \cdot 1}_{(k-1)!}}$$

$$= \frac{x^{k-1}}{(k-1)!}$$

Thus $E'(x) = E(x)$.

① $E(0) = 1$

② $\left(E(x)\cdot E(-x)\right)' = E'(x)\cdot E(-x) + E(x)\left(E(-x)\right)'$

$$= E(x)E(-x) + E(x)E'(-x)\cdot(-1)$$

$$= E(x)E(-x) - E(x)E(-x)$$

$$= 0$$

$\Longrightarrow$ $E(x)\cdot E(-x)$ is constant

$\Longrightarrow$ $E(x)\cdot E(-x) = E(0)E(-0) = 1\cdot 1 = 1 \quad \forall x\in\mathbb{C}$

Thus $E(x) \neq 0 \quad \forall x \in \mathbb{C}$ & $E(-x) = \dfrac{1}{E(x)}$.

③ Fix $a\in\mathbb{C}$ and consider the fn of x

$$\left(E(x+a)\cdot E(-x)\right)' = E(x+a)E(-x) + E(x+a)\cdot E(-x)\cdot(-1)$$

$$= 0$$

$\Longrightarrow$ $E(x+a)E(-x)$ is constant

$x=0$: constant value is $E(a)\cdot E(-0) = E(a)$

$\Longrightarrow$ $E(x+a) = E(a)\cdot\dfrac{1}{E(-x)} = E(x)\cdot E(a)$

Thus for any $a, b \in \mathbb{C}$, $E(a+b) = E(a) \cdot E(b)$.

④ $\forall a \in \mathbb{C}$, $n \in \mathbb{Z}$, $E(na) = E(a)^n$

$n=2$: $E(2a) \stackrel{?}{=} E(a)^2$

$$E(2a) = E(a+a) = E(a) \cdot E(a) = E(a)^2.$$

Easy induction shows $E(na) = E(a)^n$ for $n \in \mathbb{Z}^+$.

$$E(0 \cdot a) = E(0) = 1 = E(a)^0$$

For $n \in \mathbb{Z}^+$, $E((-n)a) = ?$

$$1 = E(0) = E(na + (-n)a) = E(na) \cdot E(-na)$$

$$1 = E(a)^n \cdot E(-na)$$

$$E(-na) = E(a)^{-n}.$$

⑤ For $a, b \in \mathbb{R}$, $E(a+bi) = E(a) \cdot E(bi)$

so to understand $E(x)$ it suffices to understand the restriction of E to $\mathbb{R}$, and to $\mathbb{R}i = \{bi \mid b \in \mathbb{R}\}$.

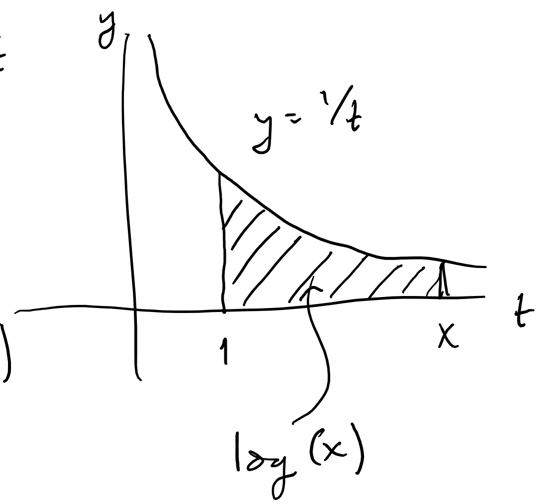
Recall $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ so if $x \in \mathbb{R}$, then $E(x) \in \mathbb{R}$.

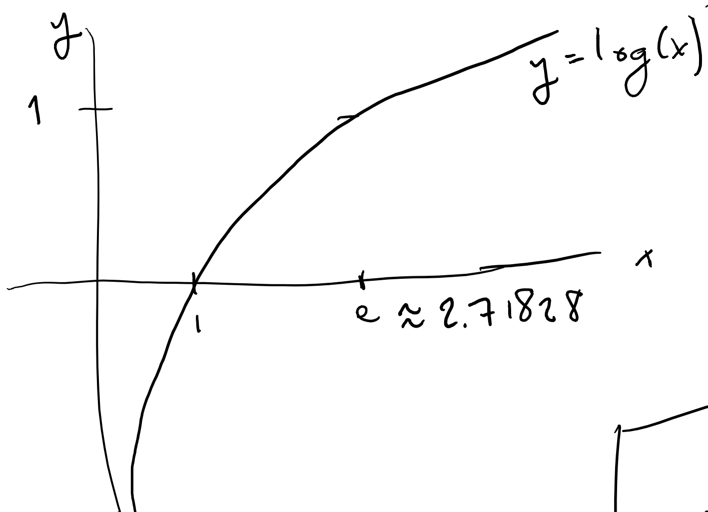
<u>Thm</u> For $x \in \mathbb{R}$, $E(x) = e^x$.

<u>Recall</u> e^x is the inverse to $\log(x)$.

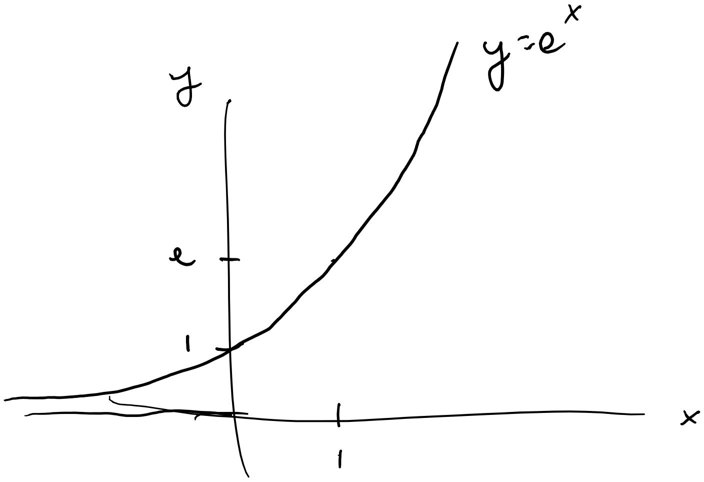
where $\log(x) = \int_1^x \frac{1}{t}\,dt$

for $x > 0$





$$\boxed{\begin{aligned} e^{\log(x)} &= x \\ \log(e^x) &= x \end{aligned}}$$



Pf Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \dfrac{E(x)}{e^x}$.

Then $$f'(x) = \frac{e^x E'(x) - E(x)\,(e^x)'}{(e^x)^2}$$

$$= \frac{e^x E(x) - E(x)\,e^x}{(e^x)^2}$$

$$= \frac{0}{(e^x)^2} = 0.$$

Thus f is constant. $f(0) = \dfrac{E(0)}{e^0} = \dfrac{1}{1} = 1$

so $f(x) = 1 \implies E(x) = e^x$. $\blacksquare$