Lecture 45

Tuesday, April 21, 2015 8:00 AM

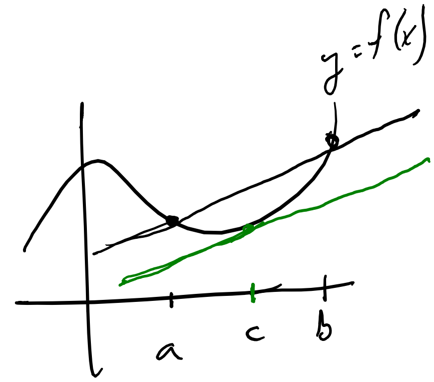
Cauchy's Mean Value Thm

$F, G : [a,b] \to \mathbb{R}$ cts, differentiable on (a,b). $\exists\, c \in (a,b)$ s.t.

$$F'(c)\cdot\big(G(b)-G(a)\big) = G'(c)\cdot\big(F(b)-F(a)\big)$$

Pf In notes. $\square$

MVT



$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Pf of Taylor's Remainder Thm

Fix $x \in B(a,r)$. Define $g : B(a,r) \to \mathbb{R}$ via

$$g(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k .$$

Then
$$g'(t) = \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}\,k(x-t)^{k-1}$$

$$= \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \cancel{\sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k} - \cancel{\sum_{\substack{\cancel{k=0}\\ k=1}}^{n} \frac{f^{(k)}(t)}{(k-1)!}}(x-t)^{k-1}$$

Note $g(x) = f(x)$, $g(a) =$ n-th Taylor poly of f at a.

Apply Cauchy's MVT to $g(t)$ & $h(t) = (x-t)^{n+1}$:

$\exists d$ between x and a s.t.

$$h'(d)\left(g(x)-g(a)\right) = g'(d)\left(h(x)-h(a)\right)$$

$$-(n+1)(x-d)^n\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-a)^k\right)$$

$$= \frac{f^{(n+1)}(d)}{n!}(x-d)^n\left(0 - (x-a)^{n+1}\right)$$

Since $d \neq x$,

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-a)^k = \frac{f^{(n+1)}(d)}{(n+1)!}(x-a)^{n+1}.$$

$\square$

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Known Taylor series:

① $$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \quad \forall x \in \mathbb{R}$$

② $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{for } x \in B(0,1)$$

log(x) power series if we understand derivatives

Derivatives of power series:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Guess $$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$$ by power rule.

This is correct!

Suppose f has radius of convergence R & $c \in B(0,R)$,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$ For $x \neq c \in B(0,R)$

$$\frac{f(x)-f(c)}{x-c} = \frac{\sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} a_k c^k}{x-c}$$

$$= \frac{\sum_{k=0}^{\infty} a_k (x^k - c^k)}{x-c}$$ [b/c the series converge]

$$= \frac{\sum_{k=0}^{\infty} a_k (x-c)(x^{k-1} + x^{k-2}c + x^{k-3}c^2 + \cdots + c^{k-1})}{x-c}$$

$$= \sum_{k=0}^{\infty} a_k (x^{k-1} + x^{k-2}c + \cdots + c^{k-1})$$

[b/c $x \neq c$]

Let $g(x) =$

Claim $g(x)$ converges on $B(0,R)$ & is cts at c.

Pf Notes — technical! $\square$

Assuming the claim, we find

$$f'(c) = \lim_{x \to c} g(x) = g(c) = \sum_{k=0}^{\infty} k a_k c^{k-1}. \quad \square$$

Thm $\sum_{k=0}^{\infty} a_k x^k$ & $\sum_{k=0}^{\infty} k a_k x^{k-1}$ have the same radius of convergence R and on $B(0,R)$,

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)' = \sum_{k=0}^{\infty} k a_k x^{k-1}.$$

Pf Remains to show radii of convergence are the same.

Follow from root test:

$$\limsup |n a_n|^{1/n} = \limsup n^{1/n} |a_n|^{1/n}$$

$$= \underbrace{\left(\limsup n^{1/n}\right)}_{=1 \text{ by } 8.2.7} \left(\limsup |a_n|^{1/n}\right)$$

$$= \limsup |a_n|^{1/n}. \quad \square$$

$$f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{on } B(0,1)$$

$$g(x) = f(-x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

<u>Note</u> $\frac{d}{dx} \log(1+x) = \frac{1}{1+x}$

Idea: Anti-differentiate

<u>Guess</u> $\log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

By ratio test, this series has radius of convergence

$$\liminf \left| \frac{(-1)^{k-1}/k}{(-1)^k/k+1} \right| = \liminf \left| \frac{k+1}{k} \right| = 1.$$

Thus $\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \right)' = \sum_{k=0}^{\infty} (-1)^k x^k$ converges on $B(0,1)$ as well.

$$= \frac{1}{1+x}.$$

$$\int \frac{dx}{1+x} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{on } B(0,1)$$

$$\parallel$$

$$\log(1+x) + C$$

Plug in $x = 0$: $C = 0$ & thus

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = \log(1+x) \quad \text{on } B(0,1).$$

Facts
- Series diverges at $x = -1$
- Series converges to $\log(2)$ at $x = 1$.