Lecture 44

Monday, April 20, 2015 8:00 AM

<u>Recall</u> Power series have radii of convergence

$$R = \liminf \left| \frac{a_k}{a_{k+1}} \right| = \limsup |a_n|^{1/n}$$

then $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for $x \in B(0, R)$

Consider $f : B(0,R) \longrightarrow \mathbb{C}$. What is it?

<u>Taylor series</u> Let f be a cpx-valued function, a in domain of f, $f^{(k)}(a)$ exists $\forall k \geq 0$.

<u>Def'n</u> The Taylor series of f centered at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k .$$

<u>Rmk</u> Suppose $f : A \to \mathbb{C}$, let $B = \{ x - a \mid x \in A \}$ let $g : B \to \mathbb{C}$, $g(x) = f(x+a)$. Then

$\sum_{k=0}^{\infty} a_k (x-a)^k$ is the Taylor series for f centered at a

$\iff \sum_{k=0}^{\infty} a_k x^k$ is the Taylor series for g centered at 0.

e.g. $f(x) = e^x$. Recall that $f'(x) = e^x$

and $f^{(k)}(x) = e^x$ so $f^{(k)}(0) = e^0 = 1$.

Thus, the Taylor series for f centered at 0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

which has radius of convergence ∞.

What does it converge to ??

Taylor's Remainder Thm

I an interval in $\mathbb{R}$, $B(a,r) \subseteq I$.

Suppose $f : I \to \mathbb{R}$ has cts $f^{(k)} : B(a,r) \to \mathbb{R}$

for $k = 1, 2, \ldots, n+1$. Then $\forall x \in B(a,r)$, $\exists\, d$ between

x and a s.t.

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(d)}{(n+1)!}(x-a)^{n+1}$$

Let's apply Taylor's Remainder Thm when $f(x) = e^x$, $I = \mathbb{R}$, $r = \infty$, $a = 0$.

For $x \in \mathbb{R}$, $\exists\, d$ b/w 0 & x s.t.

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{e^d}{(n+1)!} x^{n+1} .$$

Let $\varepsilon > 0$. Take $M > |x|$. Since $\left(\frac{1}{2}\right)^n \longrightarrow 0$ and

$$\varepsilon \cdot \frac{M!}{2^{2M-1}(3M)^M} > 0 \quad \text{so } \exists N > M \text{ s.t. if } n > N$$

$$\frac{1}{2^n} < \varepsilon \cdot \frac{M!}{2^{2M-1}(3M)^M} ,$$

Since $1 < e < 3$

$$\left| \frac{e^d}{(n+1)!} x^{n+1} \right| \le \frac{e^{|x|}}{(n+1)!} |x|^{n+1}$$

$$< \frac{3^M}{(n+1)!} M^{n+1} \quad \left[\text{since } e < 3, \; M > |x| \right]$$

$$= \frac{(3M)^M \, M^M \, M^{n+1-2M}}{M!\,(M+1)(M+2)\cdots(2M)(2M+1)\cdots n(n+1)}$$

$$= \frac{(3M)^M \quad M^M \, M^{n+1-2M}}{M!\,(M+1)(M+2)\cdots(2M)(2M+1)\cdots n(n+1)}$$

$$< \frac{(3M)^n}{M!} \quad \frac{M^{n+1-2M}}{(2M+1)(2M+2)\cdots n(n+1)} \qquad \left[\text{b/c } \frac{M}{M+k} < 1\right]$$

$$< \frac{(3M)^M}{M!} \left(\frac{1}{2}\right)^{n+1-2M} \qquad \left[\text{b/c } \frac{M}{2M+k} < \frac{1}{2}\right]$$

$$= 2^{2M-1} \frac{(3M)^M}{M!} \left(\frac{1}{2}\right)^n$$

$< \varepsilon$ [by the way we chose n].

$\Longrightarrow$ For any $x \in \mathbb{R}$ as $n \longrightarrow \infty$. $\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k} \right| \longrightarrow 0$

i.e. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

e.g. $f(x) = \log(x)$ $\quad$ log = natural log

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -x^{-2}$$

$$f'''(x) = 2x^{-3}$$

$$f^{(4)}(x) = -6x^{-4}$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\, x^{-k}$$ [to prove: induction]

Taylor series for $\log(x)$ centered at 1 : $f^{(k)}(1) = (-1)^{k-1}(k-1)!$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$$

Radius of conv : $\lim_{k\to\infty} \left| \frac{(-1)^{k-1}/k}{(-1)^k/k+1} \right| = \lim_{k\to\infty} \left| \frac{k+1}{k} \right| = 1$

Conv on $B(1,1) = (0,2)$.

If $x=0$: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-1)^k$

$$= \sum_{k=1}^{\infty} \frac{-1}{k} = -\sum_{k=1}^{\infty} \frac{1}{k} = -\infty$$

so Taylor series diverges at $x=0$.

If $x=2$: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} 1^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

converges by alt. series test!

Taylor series for $\log(x)$ converges on $(0, 2]$.