Lecture 43

Friday, April 17, 2015 8:03 AM

Exam 2 : April 24 in-class 50min — sequences, series, continuity, topology of $\mathbb{C}$

For Cauchy sequences $(a_n), (b_n)$ define an equivalence relation $(a_n) \sim (b_n)$ iff

$$\cancel{\lim_{n\to\infty} (a_n - b_n) = 0}$$

$$\lim_{n\to\infty} d(a_n, b_n) = 0$$

Power Series A power series is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k$$

where $(a_n)_{n=0}^{\infty}$ is a sequence of cpx numbers.

$x^0 = 1$ even if $x = 0$, i.e., $0^0 = 1$.

start at $k=0$, not 1

if $a_k = 0$ for $k > N$, then this is just a polynomial!

$$a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_N x^N$$

e.g. $\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$, $a_n = 1$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

$$a_n = \frac{1}{n!}$$

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} = \frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots,$$

$$a_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{1}{n} & \text{if } n \text{ odd} \end{cases}$$

Let $f = \sum a_k x^k$ and for $F = \mathbb{R}$ or $\mathbb{C}$, define

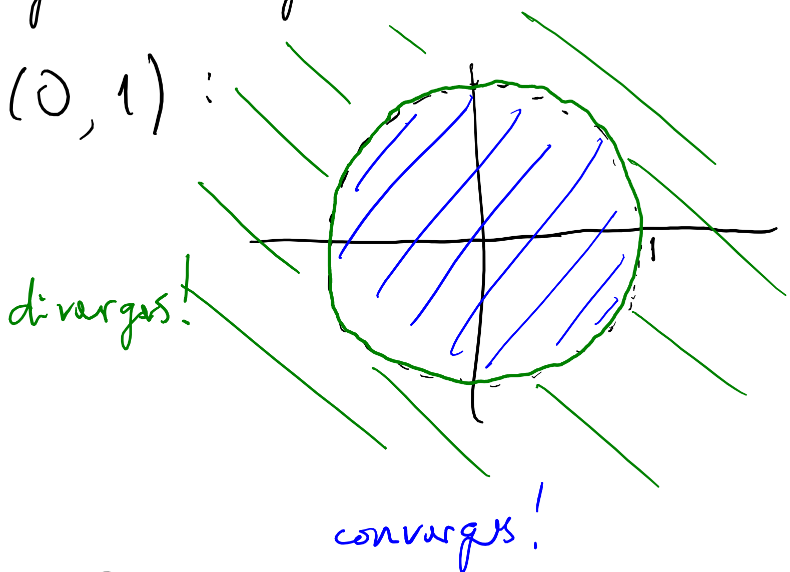
$$X_f(F) = \{ x \in F \mid \textstyle\sum a_k x^k \text{ converges} \}$$

Then f is a function

$$f : X_f(F) \longrightarrow \mathbb{C}$$
$$x \longmapsto \textstyle\sum a_k x^k$$

e.g. $f = \sum_{k=0}^{\infty} x^k$. If we substitute $r \in \mathbb{C}$ into f, we get $\sum_{k=0}^{\infty} r^k$, the geometric series for r! This converges exactly when $|r| < 1$

so $X_f(\mathbb{C}) = B(0,1)$:



Then $f : B(0,1) \longrightarrow \mathbb{C}$
$$r \longmapsto \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

<u>Thm</u> Let $f = \sum_{k=0}^{\infty} a_k x^k$ be a power series and define $\alpha = \limsup |a_n|^{1/n}$. Define
$$R = \begin{cases} 1/\alpha & \text{if } 0 < \alpha < \infty \\ 0 & \text{if } \alpha = \infty \\ \infty & \text{if } \alpha = 0 \end{cases}$$

Then $\forall x \in B(0,R) \subseteq \mathbb{C}$, f converges in $\mathbb{C}$.
If $|x| > R$, then f diverges (and $\sum |a_k x^k|$ diverges).

Recall $\limsup |a_n|^{1/n} = \inf s_m$

$$s_m = \sup\{|a_n|^{1/n} \mid n \geq m\}$$

If $\lim |a_n|^{1/n}$ exists, then $\limsup |a_n|^{1/n} = \lim |a_n|^{1/n}$.

Pf Root test for series says f converges whenever $\limsup |a_n x^n|^{1/n} < 1$ & diverges if > 1.

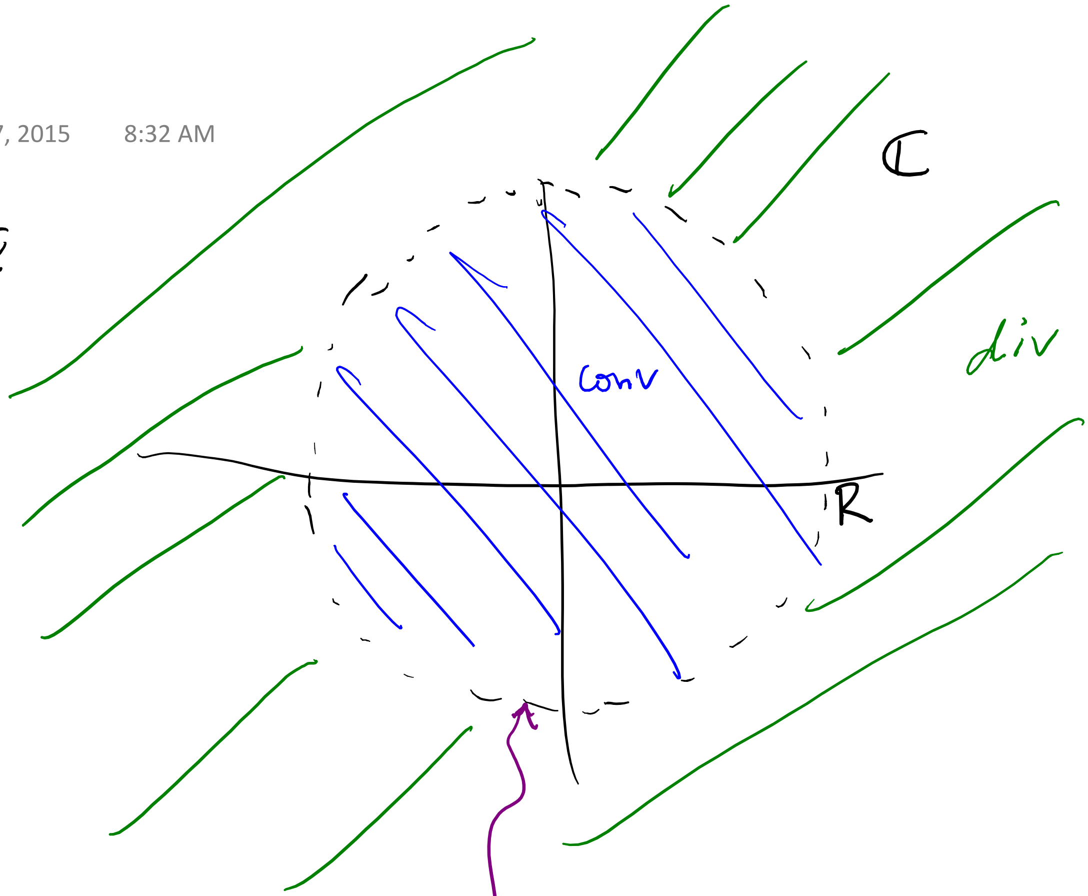
$$\limsup |a_n x^n|^{1/n} = \limsup\left(|a_n|^{1/n} |x^n|^{1/n}\right) = \limsup\left(|a_n|^{1/n}|x|\right) = |x| \underbrace{\limsup\left(|a_n|^{1/n}\right)}_{\alpha}$$

Thus, dividing by $\alpha > 0$,

f converges if $|x| < \frac{1}{\alpha} = R$

diverges if $|x| > \frac{1}{\alpha} = R$. $\square$

$R = \frac{1}{\alpha}$



if $|x| = R$, ???

Defn $R = \dfrac{1}{\limsup |a_n|^{1/n}}$ is the radius of convergence of $f = \sum a_k x^k$.

What does the ratio test tell us about R?

$f = \sum a_k x^k$ Assume $a_k \neq 0 \ \forall k$, $x \neq 0$

$$\limsup \left| \frac{a_{k+1} x^{k+1}}{a_k x^k} \right| = \limsup \left(\left| \frac{a_{k+1}}{a_k} \right| |x| \right)$$

$$= |x| \limsup \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \frac{|x|}{\liminf \left| \frac{a_k}{a_{k+1}} \right|}$$

$\liminf b_n$

$t_m = \inf \{ b_n \mid n \geq m \}$

$= \sup t_m$

$= \lim b_n$ if this exists!

So by the ratio test, f converges when

$$\frac{|x|}{\liminf \left| \frac{a_k}{a_{k+1}} \right|} < 1 \quad \text{i.e.} \quad \underbrace{|x| < \liminf \left| \frac{a_k}{a_{k+1}} \right|}_{\text{radius of convergence again!}}$$

<u>Notation</u> let $R\left(\sum a_k x^k\right)$ = radius of conv of $\sum a_k x^k$.

<u>e.g.</u> • $R\left(\sum x^k\right) = 1$

for $x \in B(0,1)$, $\sum x^k = \frac{1}{1-x}$

• $R\left(\sum k x^k\right) = \liminf \left|\frac{n}{n+1}\right| = 1$

$a_n = n$

• $R\left(\sum \frac{x^k}{k!}\right) = \liminf \left|\frac{1/n!}{1/(n+1)!}\right|$

$a_n = \frac{1}{n!}$ $= \liminf \left|\frac{(n+1)!}{n!}\right|$

$= \liminf |n+1|$

$= \infty$

So $\sum \frac{x^k}{k!}$ converges for $x \in B(0,\infty) = \mathbb{C}$!!