"Lecture" 42

Wednesday, April 15, 2015 8:13 AM

Bolzano - Weierstrass Thm Every bdd sequence in $\mathbb{C}$ has a convergent subsequence.

Recall (X, d) is a metric space if

$$d: X \times X \longrightarrow \mathbb{R} \text{ s.t.}$$

- $d(x,y) \geq 0 \quad \forall x,y \in X$
- $d(x,y) = 0$ iff $x = y \in X$
- $d(x,y) = d(y,x) \quad \forall x,y \in X$
- $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$.

A function $a: \mathbb{Z}^+ \longrightarrow X$ is a sequence in X.

Write $a = (a_n)$ where $a_n = a(n)$.

A sequence (a_n) in X converges to a limit $L = \lim_{n\to\infty} a_n \in X$ when $\forall \varepsilon > 0\ \exists N > 0$ s.t. $n > N$ implies $d(a_n, L) < \varepsilon$.

A sequence (a_n) in X is Cauchy if $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $n, m > N$ implies $d(a_n, a_m) < \varepsilon$.

Fact Every convergent sequence in X is Cauchy.

Pf As before — replace $|\ |$ w/ d. $\square$

Def'n X is Cauchy complete if every Cauchy sequence in X has a limit in X.

e.g.

- $\mathbb{R}$, $\mathbb{C}$ are Cauchy complete with respect to the metric induced by $|\ |$.
- $\mathbb{Q}$ w/ $|\ |$ metric are not Cauchy complete.

Take the decimal expansion of $\sqrt{2}$ and let a_n = truncation of this decimal of length n.

$\sqrt{2} = 1.41421356\ldots$

$a_1 = 1$ $\qquad (a_n) \longrightarrow \sqrt{2} \in \mathbb{R}$

$a_2 = 1.4$

$a_3 = 1.41$ $\qquad a_n \in \mathbb{Q}$

$a_4 = 1.414$ $\qquad a_n$ is Cauchy in $\mathbb{Q}$

$a_5 = 1.4142$ $\qquad$ but has no limit in $\mathbb{Q}$!

$\vdots$

- X a discrete metric space:

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Cauchy sequences in a discrete metric space are the eventually constant sequences, which obviously converge to the eventual constant.

- $(0,1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$

$$d(x,y) = |x-y|$$

$$\left\{ \begin{array}{l} \left(1 - \frac{1}{n}\right)_{n=2}^{\infty} \longrightarrow 1 \notin (0,1) \\ \left(\frac{1}{2^n}\right)_{n=1}^{\infty} \longrightarrow 0 \notin (0,1) \end{array} \right.$$

Cauchy sequences, but limits are <u>not</u> in $(0,1)$ so $(0,1)$ is not Cauchy complete.

Given a metric space X we can form its <u>Cauchy completion</u> X^*.

First, let $C_X = \{$ Cauchy sequences in $X \}$

For Cauchy sequences $(a_n), (b_n)$ define an equivalence relation $(a_n) \sim (b_n)$ iff

$$\lim_{n \to \infty} (a_n - b_n) = 0$$