Lecture 41

Tuesday, April 14, 2015 8:02 AM

Convergence Tests

Recall $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

Notation: $\sum_k a_k = \sum a_k$

Comparison Test (a_n) seq of real #s, (b_n) a seq of cpx #s, $\sum a_k$ converges, $a_n \geq |b_n|$. Then $\sum b_k$ converges as well.

Note Frequent comparison series: $\sum r^k$ or $\sum cr^k$ for $|r|<1$. I.e. take $a_n = r^n$.

Comparison Test for divergence: (a_n) seq of cpx #s, (b_n) seq of real #s, $b_n \geq |a_n|$ & $\sum a_k$ diverges. Then $\sum b_k$ diverges as well.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n!}$. Idea: converges via comparison w/ $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$. $n! > 2^n$ for $n \geq 4$. (check by induction)

Thus $\frac{1}{n!} \le \left(\frac{1}{2}\right)^n$ for $n \ge 4$. Thus, by the comparison test & convergence of $\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n$, we get

$\sum_{n=4}^{\infty} \frac{1}{n!}$ converges. Thus $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \sum_{n=4}^{\infty} \frac{1}{n!}$

converges!

Ratio Test (a_n) a sequence of nonzero cpx #s.

① If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum |a_n|$ & $\sum a_n$ converge.

limsup

lim inf

② If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum |a_n|$ & $\sum a_n$ diverge.

! $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cancel{=} \left| \frac{\lim a_{n+1}}{\lim a_n} \right| = \frac{0}{0}$

! If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the ratio test tells us nothing.

e.g. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$. Employ ratio test:

$$\left| \frac{5^{n+1}/(n+1)!}{5^n/n!} \right| = \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!}$$

$$= \frac{5}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$$

so the ratio test implies $\sum \frac{5^n}{n!}$ converges.

$\sum \frac{n!}{5^n}$ diverges by a similar computation.

Root Test (a_n) seq of cpx #s, $L = \lim_{n\to\infty} |a_n|^{1/n}$.

① If $L < 1$, then $\sum |a_k|$ & $\sum a_k$ converge.

② If $L > 1$, then — " — diverge.

e.g. $\sum_{n=1}^{\infty} \frac{3^{1+2n}}{n^n}$. $\left| \frac{3^{1+2n}}{n^n} \right|^{1/n} = \frac{3^{\frac{1}{n}+2}}{n}$

Note that as $n \to \infty$, $3^{\frac{1}{n}+2} \longrightarrow 9$ & $\frac{1}{n} \longrightarrow 0$,

so $\lim_{n\to\infty} \frac{3^{1/n+2}}{n} = 9 \cdot 0 = 0 < 1$ so $\sum \frac{3^{1+2n}}{n^n}$ conv.

<u>Alternating</u> <u>Series</u> <u>Test</u> If (a_n) is a nonincreasing sequence of positive #s w/ $\lim_{n\to\infty} a_n = 0$, then

$$\sum (-1)^k a_k \quad \text{converges.}$$

<u>e.g.</u> Recall that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

<u>Yet</u> by the alternating series test,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \text{ converges!}$$

In fact, $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \dots$

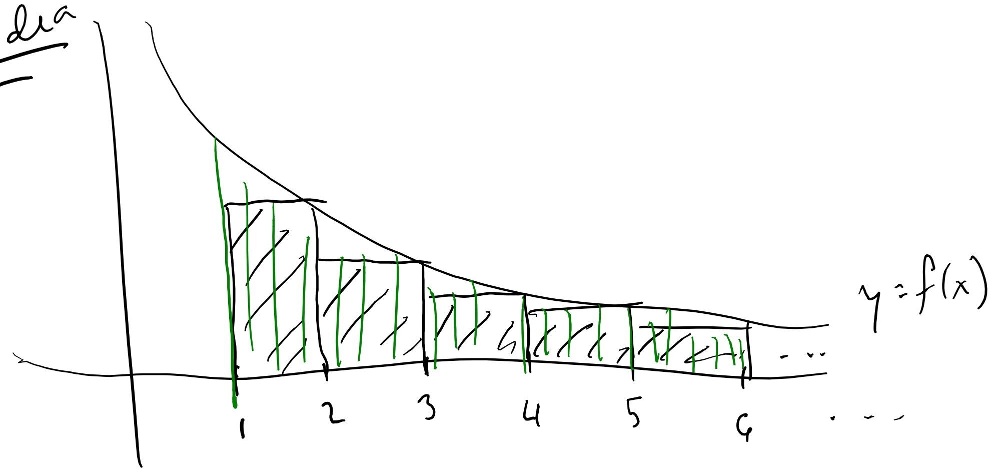
$$= \log(2)$$ [proof later]

<u>Note</u> $\sum \frac{(-1)^k}{k}$ converges, but $\sum \left|\frac{(-1)^k}{k}\right| = \sum \frac{1}{k}$ diverges.

Integral Test Let $f:[1,\infty) \longrightarrow [0,\infty)$ be a nonincreasing function s.t. $\int_1^n f$ exists $\forall n \geq 1$.

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \lim_{n\to\infty} \int_1^n f \text{ exists in } \mathbb{R}.$$

Pf Idea



$$\sum_{k=2}^{n} f(k) \leq \int_1^n f$$

Thm [p-series] $\sum_{k=1}^{\infty} k^p$ converges if $p<-1$, diverges if $p \geq -1$.

Pf If $p=-1$, then $\sum k^p = \sum \frac{1}{k}$ is the harmonic series & diverges. If $p>-1$, $n^p \geq n^{-1}$ so the comparison test implies $\sum k^p$ diverges.

Now if $p < -1$, let $f(x) = x^p$ diff'l, cts, decreasing.

$$\int_1^n f = \int_1^n x^p \, dx = \frac{x^{p+1}}{p+1} \Big|_1^n = \frac{n^{p+1} - 1}{p+1}$$

As $n \longrightarrow \infty$, $n^{p+1} \longrightarrow 0$ b/c $p+1 < 0$. Thus

$$\lim_{n \to \infty} \int_1^n f = \frac{-1}{p+1} \in \mathbb{R}$$

So the integral test implies $\sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{\infty} k^p$ conv.

<u>Note</u> $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s}$ converges for $s > 1$.

zeta : ζ

<u>Fact</u> $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$